VOEVODSKY'S SLICE FILTRATION

In 1982 Beilinson conjectured the existence of an "Atiyah-Hirzebruch" spectral sequence

$$H^{*,*}(X) \implies K_*(X)$$

starting from "motivic" cohomology (which did not exist at the time) and converging to algebraic K-theory, where X is a smooth scheme over k. This problem is hard without the right technology!

Example 1 ($X = \operatorname{Spec} k$). Using $H^{p,q}(k) = 0$ for $p > q$ and $H^{p,p}(k) = K_p^M(k)$ we can partially draw the second page. Soule-Beilinson conjecture is vanishing of $H^{p,q}$ for $p < 0$.

1. Topological realization

In order to guess how the spectral sequence above could be built, we can try to gather clues from classical homotopy theory. In fact, let $X \in Sm/\mathbb{R}$. The complex points $X(\mathbb{C})$ is a topological space with the analytic topology, and it has an action of C_2, the Galois group of the extension $\mathbb{C}/\mathbb{R}$. This yields a functor

$$t_{\mathbb{C}} \colon \mathbf{Spc}(\mathbb{R}) \to \mathcal{S}^{C_2}.$$

In fact, there is a model structure on $\mathbf{Spc}(\mathbb{R})$ for which this is a left Quillen adjoint. It is not the standard model structure, but nevertheless Quillen equivalent to the standard one. There are also two standard functors $\mathcal{S}^{C_2} \to \mathcal{S}$, given by either forgetting the C_2-action or taking C_2 fixed points.

$\mathbf{Spc}(\mathbb{R})$	$\mathcal{S}^{C_2}$	$\mathcal{S}$ (forget)	$\mathcal{S}$ (fixed points)
$S^{1,0}$	S^1	S^1	S^1
$S^{1,1}$	S^σ	S^1	S^0
$S^{p,q}$	$S^{p-q+q\sigma}$	S^p	S^{p-q}
T-spectrum	$S^{1+\sigma}$-spectrum	S^2-spectrum	S^1-spectrum
$\mathrm{Gr}_k(\mathbb{R})$	$\mathrm{Gr}_k(\mathbb{C})$	$\mathrm{Gr}_k(\mathbb{C})$	$\mathrm{Gr}_k(\mathbb{R})$
BGL	BU	BU	BO
KGL	KR	KU	KO
$H\mathbb{Z}$	$H\underline{\mathbb{Z}}$	$H\mathbb{Z}$	$H\mathbb{Z}$

This picture suggests that a good strategy might be to consider the classical Atiyah-Hirzebruch spectral sequence and port it back to the motivic world. If we knew the right spectral sequence for KR we could also try to port that back. The KR case was essentially done by Dugger, using ideas from motivic homotopy theory.

2. Classical Atiyah-Hirzebruch Spectral Sequence

Let X and Y be (topological) spectra. I want a spectral sequence converging to $[X, \Sigma^* Y] = Y^*(X)$. There are two basic approaches; to filter X or to filter Y. Analogy: To compute $\operatorname{Ext}^n(A, B)$, can take a projective resolution of A or an injective resolution of B.

2.1. Method 1: Filter X by skeleta. Assume that X is a CW-complex with n-skeleton $X^{(n)}$. There is a "tower" of cofibrations

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ X^{(n-1)} & \longrightarrow & \bigvee_{\text{(n-1)-cells}} S^{n-1} \\ \downarrow & & \\ X^{(n)} & \longrightarrow & \bigvee_{\text{n-cells}} S^{n} \\ \downarrow & & \\ \vdots & & \end{array}$$

This tower desuspends, in the sense that each vertical map is itself the cofiber of some other map; there are cofiber sequences

$$\begin{array}{ccccc} \bigvee_{(n+1)-cells} S^n & \xrightarrow{\text{attach}} & X^{(n)} & & \\ & & \downarrow & & \\ & & X^{(n+1)} & \xrightarrow{\text{collapse}} & \bigvee_{(n+1)-cells} S^{n+1}. \end{array}$$

We apply $[-, \Sigma^* Y]$ to obtain a spectral sequence from the tower. With Serre's grading convention, we get an exact couple with $A_1^{p,q} = \left[X^{(p)}, \Sigma^{p+q} Y\right]$ and

$$E_1^{p,q} = \left[\bigvee_{\text{p-cells}} S^p, \Sigma^{p+q} Y\right] \cong \left[\bigvee_{\text{p-cells}} S^0, \Sigma^q Y\right] \cong \mathrm{Hom}_{\mathbb{Z}}\left(C_p, Y^q(S^0)\right),$$

where C_p is the cellular chains in degree p. Differentials on the first page are induced by maps

$$\bigvee_{(n+1)-cells} S^n \longrightarrow X^{(n)} \longrightarrow \bigvee_{n-cells} S^n,$$

i. e. precisely maps inducing the differentials for cellular cohomology, so in sum

$$E_2^{p,q} = H^p\left(X; Y^q(S^0)\right).$$

2.2. Method 2: Filter Y using Postnikov tower. Let $\mathcal{S}$ denote the category of spectra, and

$$\mathcal{S}_{\leq n} = \{X \in \mathcal{S} : \pi_k(X) = 0 \text{ for } k > n\}.$$

The inclusion $\mathcal{S}_{\leq n} \hookrightarrow \mathcal{S}$ has a left adjoint P_n, and the unit of the adjunction $Y \to P_n Y = Y_{\leq n}$ gives a universal map from any spectrum Y to $\mathcal{S}_{\leq n}$. There is a Postnikov filtration on $\mathcal{S}$,

$$\cdots \subseteq \mathcal{S}_{\leq n} \subseteq \mathcal{S}_{\leq n+1} \subseteq \cdots \subseteq \mathcal{S}.$$

The universal property of the units thus imply that the $Y_{\leq n}$ assemble into a tower

$$\begin{array}{ccc} & & \vdots \\ & & \\ \Sigma^n H\pi_n Y & \longrightarrow & Y_{\leq n} \\ & & \downarrow \\ \Sigma^{n-1} H\pi_{n-1} Y & \longrightarrow & Y_{\leq n-1} \\ & & \downarrow \\ & & \vdots \end{array}$$

To see that the fibers have the prescribed form, consider the long exact sequence of π_* and the fact that $Y \to Y_{\leq n}$ is an isomorphism on $\pi_k, k \leq n$. Applying $[X, \Sigma^*(-)]$ to the tower yields a spectral sequence. An exact couple is given by $A_2^{p,q} = [X, \Sigma^{p+q} Y_{\leq -q}]$ and

$$E_2^{p,q} = \left[X, \Sigma^{p+q}\Sigma^{-q} H\pi_{-q} Y\right] = H^p\left(X; \pi_{-q} Y\right),$$

which equals the E_2-term from the other spectral sequence because $\pi_{-q}Y = Y^q(S^0)$.

Theorem 2 (Maunder)**.** *These spectral sequences are isomorphic.*

Proof sketch. We have already identified objects on the E_2-pages, but not differentials.

Strategy 1. One obvious strategy is to consider mapping spaces $\mathrm{Map}\left(X^{(p)}, Y_{\leq -q}\right)$. These assemble into an array of fibrations – maybe one could try a strategy similar to how one balances Ext, to use our analogy from earlier. I'm not sure how to really make this work out.

Strategy 2. The maps $Y \to Y_{\leq -q}$ and $X^{(p)} \to X$ yield maps

$$\left[X^{(p)}, \Sigma^{p+q}Y\right] \to \left[X^{(p)}, \Sigma^{p+q}Y_{\leq q}\right] \leftarrow \left[X, \Sigma^{p+q}Y\right].$$

This is a starting point to show that one has isomorphisms of exact couples. Along the way you can use isomorphisms

$$\begin{aligned} \left[X^{(n+1)}, Y_{\leq n}\right] &\cong \left[X^{(n+2)}, Y_{\leq n}\right] \cong \cdots \cong [X, Y_{\leq n}] \\ \left[X^{(n)}, Y_{\leq n}\right] &\cong \left[X^{(n)}, Y_{\leq n+1}\right] \cong \cdots \cong \left[X^{(n)}, Y\right] \end{aligned}$$

and exact sequences

$$\begin{aligned} \prod_{\text{n-cells}} \pi_n Y \to \left[X^{(n)}, Y_{\leq n}\right] &\to \left[X^{(n-1)}, Y_{\leq n}\right] \to 0 \\ H^n\left(X^{(n)}, \pi_n Y\right) \to \left[X^{(n)}, Y_{\leq n}\right] &\to \left[X^{(n)}, Y_{\leq n-1}\right] \to 0, \end{aligned}$$

all of which are easy to prove. □

Remark 3. Instead of the Postnikov tower we could also have used the Whitehead tower, given by fibers of $Y \to Y_{\leq \bullet}$. There is a "fiber sequence" of towers $Y_{>\bullet} \to Y \to Y_{\leq \bullet}$, which ought to give a long exact sequence of spectral sequences, of which one is trivial. So the spectral sequences for the Postnikov and Whitehead towers should be isomorphic (maybe with a shift).

2.3. **What have we learned?** The classical Atiyah-Hirzebruch spectral sequence is a consequence of the fact that there is a good notion of "cells", ordered by $\mathbb{Z}$, which detect weak equivalences. This is vague, but I don't know how to be more precise.

Is there any hope of applying the above constructions in motivic homotopy theory? Let X be a smooth scheme and Y be the spectrum representing algebraic K-theory. It turns out that most motivic spaces are not cellular, i. e. generated by homotopy colimits of motivic spheres $S^{p,q}$, so filtering X by a skeleton filtration does not seem like a good approach. Instead we should try to filter Y by an analogue of the Postnikov filtration. However there are some additional challenges over the classical setting:

(1) Spheres/homotopy groups are bigraded, whereas a tower needs to be indexed over integers. So there's some non-trivial choices involved in choosing how to successively change connectivity.
(2) Perhaps more importantly, identifying the "slices", i. e. analogues of the fibers of $Y_{\leq n} \to Y_{\leq n-1}$, is more difficult because motivic Eilenberg-Maclane spaces are not characterized by their homotopy groups, which is what we used in the classical setting.

3. Cellular objects

Let $\mathcal{C}$ be a stable $(\infty, 1)$-category and let A be a **set** of objects in $\mathcal{C}$.

Definition 4. The full subcategory $\langle A \rangle$ of **A-cellular** objects is the smallest one such that

(1) $A \subseteq \langle A \rangle$
(2) If X is weakly equivalent to an element of $\langle A \rangle$, then $X \in \langle A \rangle$.
(3) $\langle A \rangle$ is closed under (homotopy) colimits.

Example 5. Let $\mathcal{C} = \mathcal{S}, A = \{S^n : n \in \mathbb{Z}\}$. Then $\langle A \rangle = \mathcal{S}$ since every spectrum is weakly equivalent to a CW-spectrum.

Example 6. Let $\mathcal{C} = \mathcal{S}, A = \langle S^n : n \geq N \rangle$. Then $\langle A \rangle = \mathcal{S}_{\geq N}$, where $\mathcal{S}_{\geq N} = \{X \in \mathcal{S} : \pi_k(X) = 0 \text{ for } k < N\}$.

Recall that a T-spectrum X is a sequence of simplicial presheaves on Sm/k, $X_n, n \in \mathbb{Z}$, and bonding maps $\Sigma^{2,1} X_n \to X_{n+1}$. The category of T-spectra is denoted $\mathbf{Spt}_T(k)$.

Definition 7. The category of **cellular motivic spectra** is $\mathbf{Spt}_T(k)_c := \langle S^{p,q} : p, q \in \mathbb{Z} \rangle$.

Most motivic spectra are not cellular! (For example elliptic curves.) But some important ones are, in particular:

Proposition 8. *The algebraic K-theory spectrum $KGL = \mathbb{Z} \times BGL$ is cellular.*

Proof sketch. Since $BGL \cong \varinjlim_n \varinjlim_k \mathrm{Gr}_n(\mathbb{A}^k)$ it suffices to show that Grassmannians are cellular. Schubert cells give an affine cover of Grassmannians, but it's hard to show that finite intersections of these covers are cellular. Nevertheless it is by **DUGGER-ISAKSEN**. In the same paper they say that the question about finite intersections of Schubert cells is an open problem. □

The point of cellular spectra are that techniques and constructions from classical homotopy theory more readily carries over to this case. For example, weak equivalences between cellular motivic spectra are detected by homotopy groups $\pi_{p,q}(X) = [S^{p,q}, X]$.

Proposition 9. *Let $E \in \mathcal{C}$ be A-cellular. Suppose $[\Sigma^n S, E] = 0$ for all $S \in A$ and $n \geq 0$. Then $E \simeq *$.*

Proof. Let $\mathcal{T}$ be the class of objects $B \in \mathbf{Spt}_T(k)$ such that $\mathrm{Map}\,(B, E) \simeq *$. Then $\mathcal{T}$ is closed under weak equivalences and colimits.

Finally, the assumption implies that $A \subseteq \mathcal{T}$.This is because $\mathrm{Map}\,(\Sigma S, B) \cong \Omega\,\mathrm{Map}\,(A, B)$ in a pointed infinity category, and $\langle A \rangle$ is closed under suspensions. So $\pi_n\,(\mathrm{Map}\,(S, B)) = \pi_0\,(\mathrm{Map}\,(\Sigma^n S, B)) = 0$ for all $S \in A, n \geq 0$.

Hence $E \in \mathcal{T}$, so $\mathrm{Map}\,(E, E) \simeq *$, and in particular $\mathrm{Id}_E \simeq *$. □

Corollary 10. *Suppose $f\colon E \to F$ is a map between A-cellular objects, and $f_*\colon [\Sigma^n S, E] \to [\Sigma^n S, F]$ is an isomorphism for all $S \in A, n \geq 0$. Then f is a weak equivalence. In particular, weak equivalences between cellular motivic spectra are detected by the homotopy groups $\pi_{p,q}$.*

Proof. The cofiber is contractible by the proposition. Applying mapping spaces, we get a fiber sequence

$$* \to \mathrm{Map}\,(F, -) \to \mathrm{Map}\,(E, -)$$

In this setting weak equivalences are detected by homotopy groups, so f^* is a weak equivalence. By Yoneda's lemma f is a weak equivalence. □

4. Filtrations and spectral sequences

Theorem 11 (Lurie)**.** *Assume that $\mathcal{C}$ is (locally) presentable, and let A be a set of objects in $\mathcal{C}$. Then the inclusion $\langle A \rangle \hookrightarrow \mathcal{C}$ has a right adjoint.*

Remark 12. Voevodsky's approach used the framework of triangulated categories, working in the homotopy category. If $\mathcal{T}$ is a triangulated category and $\mathcal{L} \subseteq \mathcal{T}$ is a full triangulated subcategory, then for every $X \in \mathcal{T}$ the functor $\mathcal{L} \to \mathbf{Sets}; L \mapsto \mathrm{Hom}\,(L, X)$ is representable by an element $RX \in \mathcal{L}$ by Neeman-Brown representability. This gives a right adjoint to the inclusion. Most filtrations are not triangulated, but Voevodsky's slice filtration has the virtue of being triangulated.

Proposition 13. *Let $f\colon \mathcal{C} \to \langle A \rangle$ be right adjoint to the inclusion, and let ϵ be the counit. Then for all $X \in \mathcal{C}, S \in A, n \geq 0$, the map $\epsilon\colon fX \to X$ induces isomorphisms $[\Sigma^n S, fX] \cong [\Sigma^n S, X]$.*

Proof. This follows from the universal property of the adjunction: For any $B \in \langle A \rangle$ the counit ϵ induces an equivalence

$$\mathrm{Map}\,(B, fX) \simeq \mathrm{Map}\,(B, X)\,.$$

□

Let $\mathcal{C}$ be a presentable $(\infty, 1)$-category. Suppose

$$\cdots \subseteq A_{n-1} \subseteq A_n \subseteq \cdots \subseteq \mathcal{C}$$

be a chain of sets of objects in $\mathcal{C}$. Then

$$\cdots \langle A_{n-1} \rangle \subseteq \langle A_n \rangle \subseteq \cdots \mathcal{C}$$

is a chain of subcategories of $\mathcal{C}$. The inclusions $\langle A_n \rangle \subseteq \langle A_{n+1} \rangle$ and $\langle A_n \rangle \subseteq \mathcal{C}$ have a right adjoints W_n^{n+1} and W_n, and the diagram

$$\begin{array}{ccc} \langle A_n \rangle & \xleftarrow[W_n^{n+1}]{} & \langle A_{n+1} \rangle \\ {\scriptstyle W_n}\nwarrow & & \nearrow{\scriptstyle W_{n+1}} \\ & \mathcal{C} & \end{array}$$

commutes. The counits of the adjuctions gives for $Y \in \mathcal{C}$ rise to a "tower"

$$\cdots \to W_{n+1}Y \to W_nY \to \cdots \to Y.$$

We can map into or out of the tower to any other space X, and this gives a spectral sequence.

Example 14. Let $\mathcal{C} = \mathcal{S}$, $A_n = \{S^n\}$. Then we get the Whitehead filtration

$$\cdots \subseteq \mathcal{S}_{\geq n} \subseteq \mathcal{S}_{\geq n+1} \subseteq \cdots$$

and the Atiyah-Hirzebruch spectral sequence.

Example 15. $\mathcal{C} = \mathbf{Spt}_T(k)_c$ and $A_q = \left\{ S^{p,q'} : p \in \mathbb{Z}, q' \geq q \right\}$. This yields Voevodsky's *slice filtration.* Note that each of the categories $\langle A_n \rangle$ are triangulated. The right adjoints $\mathbf{Spt}_T(k)_c \to \langle A_q \rangle$ are denoted f_q, the cofibers $f_{q+1} \to f_q \to s_q$. Given $Y \in \mathbf{Spt}_T(k)_c$ there is a tower

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ f_{q+1}Y & \longrightarrow & s_{q+1}Y \\ \downarrow & & \\ f_qY & \longrightarrow & s_qY \\ \downarrow & & \\ \vdots & & \end{array}$$

called the *slice tower.*

Example 16. $\mathcal{C} = \mathbf{Spt}_T(k)$, instead of spheres now take $A_q = \{\Sigma^{p,q}\Sigma^\infty X_+ : p \in \mathbb{Z}, X \in Sm/k\}$. Elements of $\langle A_q \rangle$ are called q-effective. This is Voevodsky's slice filtration.

Example 17. $\mathcal{C} = \mathbf{Spt}_T(k)_c$ and $A_q = \left\{ S^{2q',q'} : q' \geq q \right\}$. This yields Spitzweck's *very effective slice* filtration. The right adjoints are denoted vf_q, the cofibers vs_q. Can do the suspension thing for non cellular spectra.

Other examples: motivic S^1-spectra, Quillen-Gersten spectral sequence, ...

5. Determination of some slices

Example 18. If S^0 is the sphere spectrum, then $s_0S^0 = H\mathbb{Z}$. This is because $H\mathbb{Z}$ vanishes in negative weights, i. e. if $q > 0$ then

$$0 = H^{p,-q}(S^0) = \pi_{-p,q}(H\mathbb{Z})$$

for all $p \in \mathbb{Z}$. So $f_1 H\mathbb{Z} \simeq *$, and thus $f_0 H\mathbb{Z} \cong s_0 H\mathbb{Z}$. So, it now suffices to show that $H\mathbb{Z}$ is 0-effective, i. e. $\in \langle A_0 \rangle$. Over fields of characteristic 0, one can prove this by expressing $H\mathbb{Z}$ as infinite symmetric products of spheres.

5.1. **Slices for KGL.** The K-theory spectrum KGL satisfies Bott periodicity

$$\Sigma^{2,1} KGL \simeq KGL,$$

which can be exploited to compute the slices.

Lemma 19. *For a cellular motivic spectrum E, we have*

(1) $\Sigma^{2,1} v f_q E \simeq v f_{q+1} \left(\Sigma^{2,1} E\right)$
(2) $\Sigma^{p,1} f_q E \simeq f_{q+1} \left(\Sigma^{p,1} E\right)$ *for all p*

and this is natural in E.

Proof. Let us prove (1); (2) is similar. The functor $\Sigma^{2,1}(-)$ maps $\{S^{2q',q'} : q' \geq q\}$ to $\{S^{2q',q'} : q' \geq q+1\}$ and preserves colimits, so it maps $\left\langle S^{2q',q'} : q' \geq q \right\rangle$ into $\left\langle S^{2q',q'} : q' \geq q+1 \right\rangle$. Hence there is a universal dashed map fitting into the commutative diagram

$$\begin{array}{ccc} \Sigma^{2,1} v f_q E & \overset{\eta}{\dashrightarrow} & v f_{q+1}\left(\Sigma^{2,1} E\right) \\ & \searrow \quad \swarrow & \\ & \Sigma^{2,1} E. & \end{array}$$

The solid maps induce isomorphisms on $\pi_{2q'+n,q'}$ for $q' \geq q+1, n \geq 0$ by Proposition 13, so η does as well. By Corollary 10 it follows that η is a weak equivalence. □

By the lemma and Bott periodicity, it follows that

$$f_q(KGL) \simeq f_q\left(\Sigma^{2q,q} KGL\right) \simeq \Sigma^{2q,q} f_0\left(KGL\right).$$

By naturality it follows that

$$s_q(KGL) \simeq \Sigma^{2q,q} s_0\left(KGL\right)$$

for all q. So we need only figure out the 0-slices.

Theorem 20. *$s_0\left(KGL\right) \simeq H\mathbb{Z}$ and consequently $s_q\left(KGL\right) \simeq \Sigma^{2q,q} H\mathbb{Z}$.*

Corollary 21. *$v s_q\left(KGL\right) \simeq \Sigma^{2q,q} H\mathbb{Z}$.*

Theorem 22. *The slice spectral sequence for KGL converges strongly.*