$$\begin{tikzcd} Y\times_X Y\times_X Y \arrow[d, shift left, "p_{12}"] \arrow[d, shift right, "p_{23}"'] \arrow[d] \\ Y\times_X Y \arrow[d, shift left, "p_1"] \arrow[d, shift right, "p_2"'] \\ Y \arrow[d, "u"] \\ X \end{tikzcd}$$

Def: (2-descent)

Assume $F: \mathcal{C}^{op} \to \mathrm{Cat}$ is a pseudo functor, $Y \xrightarrow{u} X \in \mathcal{C}$

$\mathrm{Desc}(u, F)$:

Ob: $\{ (A, \varphi: F(p_1)(A) \xrightarrow{\sim} F(p_2)(A))$, $A \in F(Y)$, s.t.:

$p_1 \circ p_{13} = p_1 \circ p_{12}$

$$\begin{tikzcd} F(p_{12})\circ F(p_1)(A) \arrow[r, phantom, "\cong"] \arrow[d, "F(p_{12})(\varphi)"'] & F(p_{13})\circ F(p_1)(A) \arrow[d, "F(p_{13})(\varphi)"] \\ F(p_{12})\circ F(p_2)(A) \arrow[d, phantom, "\wr\!\!\cong"] \arrow[r, phantom, "\text{cocycle cond.}"] & F(p_{13})\circ F(p_2)(A) \arrow[d, phantom, "\wr\!\!\cong"] \\ F(p_{23})\circ F(p_1)(A) \arrow[r, "F(p_{23})(\varphi)"] & F(p_{23})\circ F(p_2)(A) \end{tikzcd} \}$$

$p_2 \circ p_{12} = p_1 \circ p_{23}$

$p_2 \circ p_{13} = p_2 \circ p_{23}$

Mor: compatible maps

Recall: $\exists$ ss $\check{H}^r(X; \mathcal{H}^s(\mathcal{F}))$
$\Downarrow$
$H^{r+s}(X, \mathcal{F})$

- $\check{H}^0(X, \mathcal{H}^r(\mathcal{F})) = 0 \ \forall r > 0$.

$\Rightarrow$ $\mathcal{F}$ - abelian sheaf

$\Rightarrow$ $\check{H}^1(X, \mathcal{F}) \cong H^1(X, \mathcal{F})$

(need $(\mathcal{C}, \tau)$ small)

$(\mathcal{C}, \tau)$ - site.

Def: $\mathcal{G}$ - sheaf of grps on X_{et}.

$\mathcal{U} = \{U_i \to X\}$ - cover.

$\check{Z}^1(\mathcal{U}, \mathcal{G}) = \{ (g_{ij} \in \mathcal{G}(U_{ij}))_{(i,j) \in I \times I}$

s.t.

$g_{ij}|_{U_{ijk}} \cdot g_{jk}|_{U_{ijk}} = g_{ik}|_{U_{ijk}} \}$

(g'_{ij}) & (g_{ij}) are called cohomologeous if $\exists (h_i \in \mathcal{G}(U_i))_{i \in I}$ s.t.:

$g'_{ij} = h_i|_{U_{ij}} \cdot g_{ij} \cdot (h_j|_{U_{ij}})^{-1} \quad \forall i, j.$

$\check{H}^1(\mathcal{U}, \mathcal{G}) := \check{Z}^1(\mathcal{U}, \mathcal{G})/\sim$

$\check{H}^1(X, \mathcal{G}) := \varinjlim_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{G}) \leftarrow$ ptd set.

Note: If $1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 1$ is exact then

$\rightsquigarrow 1 \to \mathcal{G}'(X) \to \mathcal{G}(X) \to \mathcal{G}''(X)$

$\to \check{H}^1(X, \mathcal{G}') \to \check{H}^1(X, \mathcal{G}) \to \check{H}^1(X, \mathcal{G}'')$

Torsors: Let $\mathcal{G}$ be a sheaf of grps on X. A (right)-$\mathcal{G}$-sheaf-of-sets S is called a torsor (or a Principal Homogenous Space) if:

(1) $\exists \{U_i \to X\}$ cover s.t. $S(U_i) \neq \emptyset \ \forall i$.

(2) $\forall U \to X$, $\forall s \in S(U)$, the map

$$\mathcal{G}(U) \longrightarrow S(U) \text{ is a bijection.}$$
$$g \longmapsto s \cdot g$$

(equiv: Locally, $S \cong \mathcal{G}$ as right $\mathcal{G}$-sheaves of sets)

A torsor S is called $\boxed{\text{trivial}}$ if $S(X) \neq \emptyset$,

& is said to be trivialized by a cover $\mathcal{U} = \{U_i \to X\}$ if $S(U_i) \neq \emptyset \ \forall i$.

Notation:

$\mathrm{Tor}^{\mathcal{G}}(X)$ $\qquad$ $\mathrm{Tor}^{\mathcal{G}}_{\mathcal{U}}(X)$

Construction: $\boxed{\mathrm{Tor}^G_{\mathcal{U}}(X) \xrightarrow{\sim} \check{H}^1(\mathcal{U},G)}$

Let S - G-torsor split by $\mathcal{U}=\{U_i \to X\}$

choose: $s_i \in S(U_i)\ \forall i$.

$\exists!\ g_{ij} \in G(U_{ij})$: $s_i|_{U_{ij}} \cdot g_{ij} = s_j|_{U_{ij}}$

- this is a cocycle.
- does not depend on (s_i).

(i.e. the coho' class doesn't depend)

Prop: $\rightsquigarrow$ $\boxed{\mathrm{Tors}^G_{\mathcal{U}}(X) \xrightarrow{\sim} \check{H}^1(\mathcal{U},G)}$

(iso' classes)

Pf: • well-def: If $\alpha : S \xrightarrow{\simeq} S'$ is an iso of $\mathcal{G}$-sheaves of sets, $(\alpha(s_i))$, (s_i) give the same cocycle.

• surj: Let (g_{ij}) be a 1-cocycle for $\mathcal{G}$ rel. $\underset{\{U_i \to X\}}{\mathcal{U}}$.

$\forall V \to X$ def:

$$S(V) := \{ (g_i)_{i \in I},\ g_i \in \mathcal{G}(U_i) \mid g_i|_{V_{ij}} = g_{ij} \cdot g_j|_{V_{ij}} \quad \forall i,j \}$$

$$(V_{ij} = V \times_X U_{ij})$$

routine check:
- S is a $\mathcal{G}$-sheaf of sets
- $c(S) \underset{(?)}{=} (g_{ij})$

inj: Say $c(S) = c(S')$.

choose $s_i \in S(U_i)$, $s'_i \in S'(U_i)$

giving the same cocyle.

Def:

$$S(X) \xrightarrow{\sim} \{(g_i \in G(U_i))_{i \in I} \mid g_i h_{U_{ij}} = g'_{ij} \cdot g_{ij}\}$$

$$t \longmapsto \exists!\, g_i \quad t|_{U_i} = s'_i\, g_i$$

use uniqueness of g_i &

$\rightsquigarrow S(X) \xrightarrow{\sim} S'(X)$

$$t|_{U_i}|_{U_{ij}} = t|_{U_j} h_{U_{ij}}$$

given any $U \to X$ do the same for $S(U)$, $S'(U)$.

$\rightsquigarrow S \xrightarrow{\sim} S'$.

rk: If G is an affine gp scheme, all G-torsors for (zar $\leq \tau \leq$ fpqc) are rep.

Prop: Let zar $\leq \tau \leq \tau' \leq$ fpqc, then:

$$\boxed{\begin{tikzcd} \mathrm{Vect}_n^{\tau}(X) \arrow[r, "\sim", "(-)^{\tau'}"'] \arrow[d, "\wr"] & \mathrm{Vect}_n^{\tau'}(X) \arrow[d, "\wr"] \\ \check{H}^1(X_\tau, GL_n) \arrow[r, "\sim"] & \check{H}^1(X_{\tau'}, GL_n) \end{tikzcd}}$$

Cor: $H^1(X_{et}, \mathbb{G}_m) = \check{H}^1(X_{et}, \mathbb{G}_m)$
$= \check{H}^1(X_{zar}, \mathbb{G}_m) = \mathrm{Pic}(X)$.

<u>Verticals</u> (work $\forall \mathcal{U}$ seperately)

let $\mathcal{U} = \{u_i \to X\}$ trivializing $\mathcal{F}$

$\theta_i : \mathcal{F}|_{u_i} \xrightarrow{\sim} \mathcal{O}_{u_i}^{\oplus n}$. Def:

$g_{ij} = \theta_i|_{u_{ij}} \cdot (\theta_j|_{u_{ij}})^{-1} \in GL_n(u_{ij})$

$\rightsquigarrow \mathrm{Vect}_{n,\mathcal{U}}^{\simeq}(X) \to \underline{\check{H}^1(X_\tau, GL_n)}$

- <u>surj</u>: follows from fpqc descent for <u>objects</u> (of <u>Vect$_n$</u>)

- <u>inj</u>: $\mathrm{Vect}_n(X) \to \mathrm{Desc}(\mathrm{Vect}_n, \mathcal{U})$ is f.f. (hence conservative).

(take $\varinjlim_{\mathcal{U}}$)

$\mathrm{Vect}_n \tau \simeq \mathrm{Vect}_n \tau'$

top = horizontal: surj: If $\mathcal{F}$ is τ'-locally $\mathcal{O}_{X_{\tau'}}$-free, by fpqc descent for objects of QCoh/Coh we can construct a QCoh sheaf $\mathcal{G}$ over X, using descent for mor of sheaves we get that $\mathcal{G}^{\tau'} \simeq \mathcal{F}$. By the comm. alg. we're done, $\mathcal{G}$ is (Zar) loc. free.

(this shows surj.)

inj: It follows from 1-descent for sections of QCoh sheaves, that

$$\underbrace{(M^{\tau'})\Big|_{(e,\tau)} \equiv M}$$

$\Rightarrow$ τ-loc. free

$\mathcal{F} \simeq \mathcal{G}$ $\Leftarrow$ $\mathcal{F}^{\tau'} \simeq \mathcal{G}^{\tau'}$.

If we look at the big sites

$$\begin{array}{ccccc}
QCoh^{\tau}(X) & = & QCoh^{\tau'}(X) & \subseteq & PSh(Sch/X) \\
\cup & & \cup & & \\
Coh^{\tau}(X) & = & Coh^{\tau'}(X) & & \\
\cup & & & & \\
Vect_n^{\tau}(X) & = & Vect_n^{\tau'}(X) & &
\end{array}$$

GL_n, SL_n, Sp_{2n}

rk: G is an affine grp scheme.

then torsors $Y \to X$ are rep., &

$G \times_X Y \simeq Y \times_X Y$ as G-stuff.