Vanishing of §1 $H^*(\mathrm{Spec}\, K, \mathbb{G}_m)$ and §2 $H^*(X, \mathbb{G}_m)$ for $* \geq 3$

Galois cohom.

K/k fct. field, $\mathrm{tr\,deg}(K/k)=1$, $k=\bar{k}$

- Weil sseq
- Leray ss
- §1
- $R^q \pi_* = H^q$

X/k nonsing connected curve, $k=\bar{k}$

Recap

0) $G := \mathrm{Gal}(K^s/K)$. Then

$$\mathrm{Sh}_{\acute{e}t}(K)^{op} \longleftrightarrow \overset{\text{(continuous)}}{G\text{-sets}} = \text{discrete } G\text{-mods}$$

$$\mathcal{F} \longmapsto \mathcal{F}_{K^s}$$

↑ stalk of $\mathcal{F}$ at $\mathrm{Spec}\, K^s \to \mathrm{Spec}\, K$ geom. point

As $\mathcal{F}(\mathrm{Spec}\, K) \cong (\mathcal{F}_{K^s})^G$, ← $LHS \cong \mathrm{Hom}_{Sh}(h_K, \mathcal{F}) \cong \mathrm{Hom}_G(\{*\}, \mathcal{F}_{K^s}) \cong RHS$

have $H^q(\mathrm{Spec}\, K, \mathcal{F}) \cong H^q(G, \mathcal{F}_{K^s})$.

For $\mathcal{F} = \mathbb{G}_{m,K}$: $H^q(\mathrm{Spec}\, K, \mathbb{G}_m) \cong H^q(G, (K^s)^\times)$

1) $H^1(X, \mathbb{G}_m) \cong \mathrm{Pic}(X)$ $\forall X$ scheme

2) $H^2(\mathrm{Spec}\, K, \mathbb{G}_m) \underset{\text{Stephan}}{\cong} \mathrm{Br}(K) \underset{\text{Adam via Tsen}}{\cong} 0$ ← if K is C^1

FACT I

$X_\infty = \lim_{i \in I} X_i$, (X_i, I) inverse system

X_i qcqs., $X_i \xrightarrow{f_{ij}} X_j$ affine, denote $X_i \xrightarrow{f_i} X_\infty$

$\mathcal{F}_i \in \mathrm{Sh}(X_{i,\acute{e}t})$ w/ $f_{ij}^{-1}\mathcal{F}_j \to \mathcal{F}_i$ compatibly

$\mathcal{F}_\infty = \mathrm{colim}_I\, f_i^{-1} \mathcal{F}_i$

$\Rightarrow H^p(X_\infty, \mathcal{F}_\infty) \cong \mathrm{colim}_I\, H^p(X_i, \mathcal{F}_i)$

FACT II:

$f: X \to Y \Rightarrow (R^q f_* \mathcal{F})_{\bar{y}} = H^q(\underbrace{X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,\bar{y}})}_{=: \tilde{X}}, \mathcal{F})$

(Idea: use lim over ét. nbhds)

Cor: $f: X \to Y$ finite $\Rightarrow R^q f_* \mathcal{F} = 0$ for $q > 0$

Pf: $\tilde{X}$ is $\mathrm{Spec}(A)$, A st. hens. $\Rightarrow A = \prod A_i$, A_i st. hens. local

$H^q(\mathrm{Spec}\, A_i, \mathcal{F}) = 0$.

§1

Prop: $K = K^s = \bar{K}$; $G = \mathrm{Gal}(\bar{K}/k)$

Galois H^* version

Assume $\mathrm{Br}(K') = 0$ $\forall K'/K$ finite

Then

(1) $H^q(G, M) = 0$ for M torsion G-mod, $q \geq 2$

(2) $H^q(G, \bar{K}^\times) = 0$ for $q \geq 1$

[Serre] Galois cohom.

[Milne] via CFT

Étale H^* version

Assume $H^2(\mathrm{Spec}\, K', \mathbb{G}_m) = 0$ $\forall K'/K$ finite

(1') $H^q(\mathrm{Spec}\, K, \mathcal{F}) = 0$ $\forall \mathcal{F}$ torsion sheaf, $q \geq 2$

(2') $H^q(\mathrm{Spec}\, K, \mathbb{G}_m) = 0$ $\forall q \geq 1$

[Stacks] Today.

Cor: $k = \bar{k}$, $\mathrm{tr\,deg}(K/k) = 1$.

$\Rightarrow H^q(\mathrm{Spec}\, K, \mathbb{G}_m) = 0$ $\forall q \geq 1$.

Pf of Cor: By (2') of Prop, WTS $\forall K'/K$ finite, $\mathrm{Br}(K') = 0$.

This (almost) follows from Tsen

(take colim over $k \subset \underset{\substack{\text{fin. gen.}\\ \text{tr deg}=1}}{k'} \subset K'$, so $\mathrm{Br}(K') = \mathrm{colim}\, \mathrm{Br}(k')$)

Pf of Prop:

step 1 (1') ⇒ (2'):

$H^1(\mathrm{Spec}\, K, \mathbb{G}_m) = \mathrm{Pic}(\mathrm{Spec}\, K) = 0$

$H^2(\mathrm{Spec}\, K, \mathbb{G}_m) = \mathrm{Br}(K) = 0$

Now,

$$0 \to \overset{A_1}{\ker(\varphi)} \to \bar{K}^\times \xrightarrow{\varphi} \underset{\mathbb{Z}[G]}{\overset{A_3}{\bar{K}^\times \otimes \mathbb{Q}}} \to \overset{A_4}{\mathrm{coker}(\varphi)} \to 0 \qquad (*)$$

- $H^{q \geq 2}(G, A_i) = 0$
  - ker, coker (φ) are torsion, so (1) applies

    (eg, $k \otimes \mathbb{Q}/\mathbb{Z} = k^a \otimes 1/b \sim 1 \otimes 1/b$ in coker)

  - $H^{q \geq 1}(G, \bar{K}^\times \otimes \mathbb{Q}) = 0$ as $\bar{K} \otimes \mathbb{Q}$ is $\mathbb{Q}$ v.sp.

    (idea: H^i is both torsion (true for any profinite $H^i(G, M)$) and a $\mathbb{Q}$-v.s.)

    (Stacks 0DV3)

- Split $(*)$ into s.e. seqs. and use their LES's and $H^{q \geq 2}(G, A_i) = 0$.

step 2 (1): "get higher $H^q(\mathcal{F}) = 0$ from $H^2 = 0$"

Lemma: ℓ prime, $n > 0$ (don't need $K^s = \overline{K}$, use $G = \mathrm{Gal}(K^s/K)$)

$H \subset G$ max. pro-ℓ sub gp

L/K correspo. field extension

Then :

$$\begin{tikzcd}
H^q(\mathrm{Spec}\, K, \mathcal{F}) = 0 \ \forall_{\mathcal{F}\ \ell\text{-power torsion}}^{q \geq n} \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow, "\checkmark" ', "(A)"] & H^n(\mathrm{Spec}\, L, \underline{\mathbb{Z}/\ell}) = 0 \arrow[d, Leftrightarrow, "\checkmark\ (C)"] \\
H^n(\mathrm{Spec}\, K, \mathcal{F}) = 0 \ \forall \mathcal{F}\ \ell\text{-torsion} \arrow[r, Leftrightarrow, "(B)" '] & H^n(\mathrm{Spec}\, L, \mathcal{F}) = 0 \ \forall \mathcal{F}\ \ell\text{-torsion}
\end{tikzcd}$$

Pf; A) Induction:

Let $\mathcal{F}$ any ℓ-torsion sheaf., WTS $H^{n+1}(K, \mathcal{F}) = 0$

Let $g : \mathrm{Spec}\, K^s \longrightarrow \mathrm{Spec}\, K$

$\rightsquigarrow \quad 0 \to \mathcal{F} \xrightarrow{\text{res}} g_* g^{-1} \mathcal{F} \longrightarrow \underbrace{g_* g^{-1}\mathcal{F}/\mathcal{F}}_{\ell\text{-torsion, so } H^n(\mathrm{Spec}\, K, -\text{''}-) = 0} \longrightarrow 0$ exact

By LES, WTS $H^{n+1}(\mathrm{Spec}\, K, g_* g^{-1} \mathcal{F}) = 0$ ☆

$g = \lim\limits_{I} g_i$, where g_i finite étale

$\Rightarrow R^{>0} g_{i,*} = 0 \xRightarrow{[\text{Stacks } 09Z1]} R^{>0} g_* = 0$

$\xRightarrow{\text{Leray ss}} H^{n+1}(\mathrm{Spec}\, K, g_* g^{-1}\mathcal{F}) = H^{n+1}(\mathrm{Spec}\, K^s, g^{-1}\mathcal{F}) = H^{n+1}(\underbrace{\mathrm{Gal}(K^s/K^s)}_{=0}, -) = 0$

(B) Let $g : \mathrm{Spec}\, L \to \mathrm{Spec}\, K$

Idea:
$$\begin{tikzcd}
Sh(K_{\acute{e}t}) \arrow[r, bend left, "g^{-1}"] & Sh(L_{\acute{e}t}) \arrow[l, bend left, "g_*"]
\end{tikzcd}$$

These functors send ℓ-torsion sheaves with $H^n \neq 0$
to ——— " ———

g^{-1}: $L = \bigcup L_i$ where L_i/K finite, $\mathrm{Spec}\, L_i \xrightarrow{g_i} \mathrm{Spec}\, K$.

$\xrightarrow{\text{FACT I}} H^n(\mathrm{Spec}\, L, g^{-1}\mathcal{F}) = \operatorname{colim}\limits_{I} H^n(\mathrm{Spec}\, L_i, g_i^{-1}\mathcal{F})$

But by "méthode de la trace" uses that $\deg(g_i)$ coprime to ℓ

$H^n(\mathrm{Spec}\, K, \mathcal{F}) \hookrightarrow H^n(\mathrm{Spec}\, L_i, g_i^{-1}\mathcal{F})$ compatibly, so $\operatorname{colim}\limits_{I} \neq 0$

g_*: $g_* = \lim g_i$ & $R^{>0} g_{i*} = 0$

$\Rightarrow R^{>0} g_* = 0 \xRightarrow{\text{Leray ss.}} H^n(\mathrm{Spec}\, K, g_* \mathcal{G}) = H^n(\mathrm{Spec}\, L, \mathcal{G}) \neq 0$

(C) $\mathcal{F} = \operatorname{colim} \mathcal{F}_i$ with $\mathcal{F}_i$ finite loc. constant

$\mathcal{F} \longleftrightarrow M \in G\text{-mod}$
Consider any $\{x_1, \dots, x_m\} \in M$
$M' := \mathrm{span}_{G\text{-mod}}\{x_1, \dots, x_m\}$ is finite
Then M is colim of such $M' \subset M$

WTS $H^n(G, M) = 0$ G pro ℓ-gp., M finite ℓ-torsion

assuming $H^n(L, \underline{\mathbb{Z}/\ell}) = 0$

There exists G-invariant filtration

$0 = M_0 \subset \dots \subset M_r = M$

w/successive quotients $M_{i+1}/M_i \cong \mathbb{Z}/\ell$ w/trivial G-action

$\rightsquigarrow 0 = \mathcal{F}_0 \subset \dots \subset \mathcal{F}_r = \mathcal{F}$

w/succ. quot. $\mathbb{Z}/\ell$, so $H^n(\operatorname{Spec} L, \mathbb{Z}/\ell)=0$ & LES in cohomology give the result.

step 3 (1); analyze at each prime

By Fact I, WTS $\forall \ell$ prime, $H^q(\operatorname{Spec} K, \mathcal{F})=0$ $\forall q\geq 2$, $\mathcal{F}$ ℓ-torsion

With $H \subset G$ and L/K as in lemma, WTS $H^2(\operatorname{Spec} L, \underline{\mathbb{Z}/\ell\mathbb{Z}})=0$

max pro-ℓ gp.

- $\ell \neq p$:

$L = \bigcup L'$ where L'/K finite

$$0 \to \mu_{\ell, L'} \to \mathbb{G}_{m,L'} \xrightarrow{(\ )^\ell} \mathbb{G}_{m,L'} \to 0 .$$

- $$H^q(\operatorname{Spec} L', \mathbb{G}_{m,L'}) = \begin{cases} 0 & q=1 \quad \text{by Pic} \\ 0 & q=2 \quad \text{by } Br(L')=0 \text{ assumption} \end{cases}$$

$\Rightarrow H^q(\operatorname{Spec} L', \mu_\ell)=0$

Fact I $\Rightarrow H^q(\operatorname{Spec} L, \mu_\ell)=0$

and $\mu_\ell = \underline{\mathbb{Z}/\ell}$ (else get $L \subset L''$ Galois, deg coprime to ℓ)

contradicting defn. of L

adjoining $\mu_\ell(\bar{L})$ to L.

- $\underline{\ell = p}$:

(Artin-Schreier) $0 \to \underline{\mathbb{Z}/p\mathbb{Z}} \to \mathbb{G}_{a,L} \xrightarrow{x \mapsto x^p - x} \mathbb{G}_{a,L} \to 0$

For char $L=p$, which holds for L/K

$$H^q(L, \mathbb{G}_{a,L}) = H^q(\operatorname{Spec} L, \mathcal{O}_L) = 0 \quad \text{for } q \geq 1$$

(Stacks 03P6)

vanishing of higher cohom on affines for structure sheaf

Hence, by LES of A-S. short ex. seq.

$$H^2(\operatorname{Spec} L, \mathbb{Z}/p) = 0$$

□

§2

Thm: $k=\bar{k}$, X/k non sing, conn. curve

$$\Rightarrow \quad H^q(X,\mathbb{G}_m) = \begin{cases} k^\times & q=0 \quad ✓ \\ \mathrm{Pic}(X) & q=1 \quad ✓ \\ 0 & q\geq 2 \end{cases}$$

Pf: Use LES of Weil:, $g:\eta\to X$ gen. pt.

$$0\to\mathbb{G}_m\to g_*\mathbb{G}_{m,\eta}\to\bigoplus_{\substack{z\to X\\ \text{closed}}} i_{z*}\mathbb{Z}_z\to 0$$

Lemma: $R^q g_*\mathbb{G}_{m,\eta}=0$.

Pf: For any $x\to X$ closed, let $K:=\mathrm{Frac}(\mathcal{O}^h_{X,x})$. Have:

$$\operatorname{Spec}K = \eta\times_X \operatorname{Spec}(\mathcal{O}^h_{X,x})$$

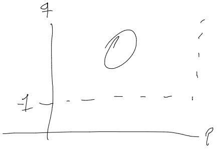
$$\overset{\text{FACT II}}{\Longrightarrow} (R^q g_*\mathbb{G}_{m,\eta})_x = H^q(\operatorname{Spec}K,\mathbb{G}_m) \underset{\substack{\uparrow\\ \text{Thm 1}}}{=} 0$$

$\left(K/k(X) \text{ algebraic} \Rightarrow \operatorname{tr\,deg}(K/k)=1\right)$

- $H^{q>0}(X, g_*\mathbb{G}_{m,\eta})=0$

Leray: $E_2^{p,q}=H^p(X,R^q g_*\mathbb{G}_{m,\eta})\Rightarrow H^{p+q}(\eta,\mathbb{G}_{m,\eta})$

By lemma: E_2:



$$\Rightarrow H^q(X, g_* \mathbb{G}_{m,\eta}) \simeq H^q(\eta, \mathbb{G}_{m,\eta}) \underset{\substack{\uparrow \\ \text{Thm } 1}}{=} 0$$

$$\cdot\ H^{q>0}\Big(X, \bigoplus_z i_{z*} \mathbb{Z}_z\Big) = 0$$

$\underline{\text{FACT 1}} \Longrightarrow$ deal w/each $z \to X$

i_z finite $\Rightarrow R^q i_{z*} = 0$ for $q>0$

Leray $\Longrightarrow H^p(X, i_{z*}\mathbb{Z}_z) \simeq H^p(z, \mathbb{Z}_z) = 0$

$\uparrow z = \operatorname{Spec}(k)$

$k = \bar{k}$; so $\operatorname{Gal}(k^s/k) = 0$