Tsen's Theorem: Suppose X is an (irreducible) curve over an algebraically closed field k. Then the function field K is C_1.

Proof: We can reduce to the case that X is projective and non-singular by passing to the projective closure of some open subset and then normalizing. (These all preserve the function field).

The function field is given by rational functions on X, and we want to stratify the rational functions by the order of their poles along an ample divisor.
Namely, we can pick a point of codimension 1 $v \in X$, and define a family of line bundles

$$\mathcal{O}_X(rv)$$

where the sections are given by

$$\mathcal{O}_X(rv)(U) = \{ f \in K \mid f = 0 \text{ or } \mathrm{ord}_v(f) + rv \geq 0 \}$$

In particular, sections of $\mathcal{O}_X(rv)$ consist of those rational functions f with poles of order at most r at v.

Now fix a homogeneous degree d polynomial F in $n > d$ variables. We must construct a non-trivial root. We can view F as inducing a map:

$$\Gamma(X, \mathcal{O}(rv))^n \xrightarrow{F} \Gamma(X, \mathcal{O}(drv))$$

$$(g_1, \ldots, g_n) \longmapsto F(g_1, \ldots, g_n)$$

Let $a_r = \dim_k \Gamma(X, \mathcal{O}(rv))$ and $b_r = \dim \Gamma(X, \mathcal{O}(drv))$

We have b_r degree d homogeneous polynomials on a vector space of dimension na_r and we want to find a common root. Of course, this is guaranteed if

$$na_r > b_r$$

Luckily, there's a big powerful theorem that tells us the dimension of these vector spaces: Riemann Roch

One version of RR states that for a non-singular projective curve X over an alg closed field, and D a divisor

$$\dim(H^0(X, \mathcal{O}(D))) - \dim(H^0(X, \mathcal{O}(K_X - D))) = \deg(D) - g + 1$$

where K_X is the canonical divisor.

Now if r is sufficiently large $\mathcal{O}(K - rv)$ has no global sections so we get

genus

$$\dim\Gamma(X, \mathcal{O}(rv)) = r - g + 1$$
$$\dim\Gamma(X, \mathcal{O}(rdv)) = dr - g + 1$$

So we can choose r s.t.

$$n(r - g + 1) > dr - g + 1$$

which is exactly what we needed.

$\square$

Why? (only defined up to linear equivalence)

K_X is constructed as follows: for ω a rational n-form,

$K = \text{zeros}(\omega) - \text{poles}(\omega)$

For sufficiently large r

$\deg(K_X - rv) < 0$

So if $f \in \mathcal{O}(K_X - rv)$ were non-zero

$\deg(D + \operatorname{div} f) \geq 0$

but

$\deg(D + \operatorname{div} f) = \deg(D)$

since $\deg(\operatorname{div} f) = 0$ $\forall$ rational functions f

---

Background: For a Weil divisor

$$D = \sum \lambda_v v$$

the degree of D is defined to be

$$\deg(D) = \sum \lambda_v$$

Fact: If D is a principal divisor – i.e.

$$D = \operatorname{div} f$$

for some $f \in K(X)$, then $\deg(D) = 0$. (at least on projective varieties)

Thus if $\deg(D) < 0$, $\dim(\Gamma(X, \mathcal{O}(D))) = 0$.

[For X a projective curve over a field, $\chi(\mathcal{O}_X(D)) = \deg D + \chi(\mathcal{O}_X)$. For a principal divisor $\operatorname{div}(f)$, $\mathcal{O}(\operatorname{div}(f))$ is isomorphic to $\mathcal{O}_X$]

More on The Brauer Group:

Stefan asked about 3 basic bits of background that I glossed over last time.

(i) Why is the base change of CSA a CSA? — This is easy

(ii) Why do Galois splitting fields exist? — This isn't hard, but is more involved than I want to get into

(iii) Why is $Br(k) \simeq H^2(\operatorname{Spec} k_{et}, \mathbb{G}_m)$? — This is a little tricky, and I'll give a sketch of it.

(i) Ok, there are two things to show - centrality + simplicity. Simplicity is the hard part, so lets save it.

Lemma: Let A, A' be two algebras over k, $B \subset A$, $B' \subset A'$ two subalgebras with centralizers C and C' respectively. Then the centralizer of $B \otimes_k B'$ is $C \otimes_k C'$.

Proof: Let $\hat{C}$ be the centralizer of $B \otimes_k B'$ in $A \otimes_k A'$.

To show $C \otimes_k C' \subset \hat{C}$ just check on simple tensors

$$(c \otimes c')(b \otimes b') = cb \otimes c'b' = bc \otimes b'c' = (b \otimes b')(c \otimes c')$$

On the other hand, $\hat{C}$ commutes with $B \otimes_k 1$, and thus is contained in $C \otimes_k A'$. Similarly, $\hat{C} \subset A \otimes_k C'$ so

$$\hat{C} \subset C \otimes_k A' \cap A \otimes_k C' = C \otimes_k C'.$$

$\square$

In particular, if B and B' are the centers of A and A', the center of $A \otimes_k A'$ is $B \otimes_k B'$.

Ok, so this tells us if A is CSA/k and k'/k is a field extension, $A \otimes_k k'$ is central. We need simplicity! The idea here is to appeal to Wedderburn & reduce to the case where A is a division algebra.

roughly, logic is: A simple $\Longleftrightarrow$ $\mathrm{Mat}_{n\times n}(A)$ simple

A simple $\Longrightarrow$ $A \otimes_k D$ simple for any division ring.

Lemma: Every two sided ideal $J \leq \mathrm{Mat}_{n\times n}(A)$ is of the form

$$I \,\mathrm{Mat}_{n\times n}(A)$$

for $I \leq A$ a two sided ideal.

Proof. Just decompose J

$$J = \bigoplus_{i,j} e_{ii} J e_{jj}$$

Elementary row operations give isomorphisms between all the summands and each summand is a two sided ideal of A.

$\square$

Lemma: Let V be a k-vector space and D a central division algebra over k. Let $W < V \otimes_k D$ be a two-sided D-sub vector space. Then W is generated over D by $W \cap (V \otimes 1)$

Proof: Let $V' = \{v \in V \mid v \otimes 1 \in W\}$ so $V' \otimes_k D \subset W$ and

$$W / V' \otimes_k D \subset (V/V') \otimes_k D$$

If $\bar{v} \in V/V'$ is non-zero, and $\bar{v} \otimes 1 \in W/V' \otimes_k D$ then $v \otimes 1 \in W$ for any lift v of $\bar{v}$ to V. Of course, this contradicts the definition of V', so it suffices to replace W with $W/V' \otimes_k D$ and V with V/V' and show that

$$V \otimes 1 \cap W \neq 0 \quad \text{if } W \neq 0.$$

PROOF IS WRONG
See Stacks tag 074B

To see this, let $w \in W$ and write

$$w = \sum_{i=1}^{n} v_i \otimes d_i$$

where n is minimal. Right multiplication by d_1^{-1} gives us a new element

$$\hat{w} = \hat{v}_1 \otimes 1 + \sum_{i=2}^{n} \hat{v}_i \otimes \hat{d}_i$$

If $n=1$, we're done. If $n>1$, then for any $c \in D$ we have

$$\hat{w}c - c\hat{w} = \sum_{i=2}^{n} \hat{v}_i \otimes (c\hat{d}_i - \hat{d}_i c)$$

and we're done by induction.

□

Corollary: Let D be a central division algebra and A an algebra. Then any two sided ideal $J \subseteq A \otimes_k D$ is of the form $I \otimes_k D$ for $I \subseteq A$ a two sided ideal.

Proof: Define $I := \{ a \in A \mid a \otimes 1 \in J \}$ and apply the previous lemma.

$\square$

Ok, finally

Theorem: If A is a CSA/k and k'/k is a field extension, then $A \otimes_k k'$ is a CSA/k'.

Proof: $$A \otimes_k k' \underset{\substack{\uparrow \\ \text{Artin} \\ \text{Wedderburn}}}{\simeq} \mathrm{Mat}_{n\times n}(D) \otimes_k k' \simeq \mathrm{Mat}_{n \times n}(D \otimes_k k')$$

We know $A \otimes_k k'$ is central, and is simple if and only if $D \otimes_k k'$ is. But D is a central division algebra and k' is simple, so $D \otimes_k k'$ is simple.

$\square$

So CSAs are stable under base change!