ETALE READING GROUP EXERCISES

THE ETALE BOYS

Exercise 1. Let $f\colon X \to Y$ be a separated morphism of schemes with X irreducible and Y reduced which is locally an isomorphism, i. e. for all $x \in X$ there is an open neighborhood $U \subseteq X$ of x such that $f|_U\colon U \to f(U)$ is an isomorphism onto an open subset $f(U) \subseteq Y$. Show that f is an open embedding.

Solution. First, let us take X to be separated. Without loss of generality we may assume that f is surjective; otherwise, replace Y by $f(U)$.

Let $\{U_i\}_i$ be an open cover of X for which f restricts to isomorphisms $f_i\colon U_i \to V_i$, where $V_i \subseteq Y$. Let $g_i\colon V_i \to U_i$ be the inverse of f_i. We will show that the g_i's glue to a global inverse of f, i. e. that $g_i|_{V_i \cap V_j} = g_j|_{V_i \cap V_j}$.

Since X is irreducible, $U_i \cap U_j \neq \emptyset$ for all i, j, and f maps this isomorphically onto an open subset $f(U_i \cap U_j) \subseteq V_i \cap V_j$. It is now clear that

$$(g_i|_{V_i \cap V_j})|_{f(U_i \cap U_j)} = (g_j|_{V_i \cap V_j})|_{f(U_i \cap U_j)}.$$

Because non-empty open subsets of irreducible spaces are irreducible, each U_i and consequently V_i is irreducible, and $V_i \cap V_j$ is also irreducible. Therefore $f(U_i \cap U_j)$ is dense in $V_i \cap V_j$. By the Reduced-to-Separated Theorem [Vak, Theorem 10.2.2] it follows that $g_i|_{V_i \cap V_j} = g_j|_{V_i \cap V_j}$.

To see that the claim is true when f instead of X is separated we cover X by affine opens first. Since affine schemes are separated and separated morphisms preserve separated schemes under pullback, we can now apply the above strategy to each affine open in our chosen cover. □

Exercise 2. Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms of schemes with g formally unramified and $g \circ f$ formally etale. Show that f is formally etale.

Solution. In the commutative diagram

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ T' \arrow[r, "\beta"] \arrow[ur, dashed] & Y, \end{tikzcd}$$

where $T \to T'$ is a nilpotent thickening, we must show that the dashed arrow exists and that there is only one such arrow. Extend the diagram:

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "f"] \\ T' \arrow[r, "\beta"] \arrow[d, equal] & Y \arrow[d, "g"] \\ T' \arrow[r, "g\circ\beta"] \arrow[uur, dashed, "\lambda"] & Z. \end{tikzcd}$$

Since gf is formally smooth, the long dashed arrow λ exists, i. e. there exists $\lambda\colon T' \to X$ with $gf\lambda = g\beta$ and $\lambda j = \alpha$. But then $f\lambda$ and β both fit in as dashed arrows in the diagram

$$\begin{tikzcd} T' \arrow[r, "\beta"] \arrow[d, equal] & Y \arrow[d, "g"] \\ T' \arrow[r, "g\beta"] \arrow[ur, dashed] & Z, \end{tikzcd}$$

so because g is formally unramified, $f\lambda = \beta$. Therefore λ fits in as a dashed arrow in the original diagram.

It remains to show that λ is unique with this property. Suppose λ' is another such morphism. Then both λ and λ' fit in as dashed arrows in the diagram

$$\begin{tikzcd} T \arrow[r, "\alpha"] \arrow[d, "j"'] & X \arrow[d, "gf"] \\ T' \arrow[r, "g\beta"] \arrow[ur, dashed] & Z, \end{tikzcd}$$

so since gf is formally unramified it follows that $\lambda = \lambda'$. □

Exercise 3. A topological space X is called *sober* if every non-empty irreducible closed subset $A \subseteq X$ is the closure of a unique point $x \in X$. Show that the topology of a sober space X can be recovered from the associated site $\mathrm{Open}(X)$.

Solution. Let X be a proper space, and consider the site $\mathrm{Closed}(X) = \mathrm{Open}(X)^{op}$, in which objects are closed subsets of X and morphisms are inclusions of subsets. Call an element $A \in \mathrm{Closed}(X)$ *reducible* if there exists non-initial objects $A \neq B_1, B_2 \in \mathrm{Closed}(X)$ such that $A = B_1 \vee B_2$, and *irreducible* if it is not reducible. Define a set

$$\tilde{X} = \{A \in \mathrm{Closed}(X) : A \text{ is irreducible}\}.$$

Define a topology by calling a subset $S \subseteq \tilde{X}$ *closed* if every irreducible $A \hookrightarrow \vee S$ is an element of S.

Claim 1. Let S be a collection objects in $\mathrm{Closed}(X)$. Then $\vee S = \overline{\bigcup_{A \in S} A}$, where the symbol on the left hand side is the join of S.

From the claim it follows that $A \subseteq X$ is irreducible if and only if A is irreducible as an element of $\mathrm{Closed}(X)$. From this and the definition of a sober space it follows that

$$\begin{aligned} \varphi\colon X &\to \tilde{X} \\ x &\mapsto \overline{\{x\}} \end{aligned}$$

is a bijection.

It remains to show that $V \subseteq X$ is closed if and only if $\varphi(V)$ is closed. Note that by the claim we have in particular

$$\vee\varphi(V) = \overline{V}.$$

Assume first that V is closed. If $\overline{\{x\}} \hookrightarrow \vee\varphi(V) = V$, then $x \in V$, so $\overline{\{x\}} \in \varphi(V)$. Conversely, assume that $\varphi(V)$ is closed. If $x \in \overline{V}$, then $\overline{\{x\}} \hookrightarrow \vee\varphi(V)$, so $\overline{\{x\}} \in \varphi(V)$ (universal property of the join), hence $x \in V$.

Proof of Claim: Let S be a collection of objects in $\mathrm{Closed}(X)$. Suppose $V \in \mathrm{Closed}(X)$ is an element with $A \hookrightarrow V$ for every $A \in S$. Then $\overline{\bigcup_{A \in S} A} \hookrightarrow V$. This is the universal property of $\vee S$. □

Exercise 4. Show that there are spaces that cannot be recovered from $\mathrm{Open}(X)$.

Solution. Let X be a set with more than one point endowed with the trivial topology. Then $\mathrm{Open}(X) = \{\emptyset \hookrightarrow X\}$. Clearly the set X cannot be recovered from this site. □

Exercise 5.
(1) Let Z be a G-set. Show that the representable functor $h_Z = \mathrm{Hom}_G(-, Z)$ is a sheaf in T_G.
(2) Finish the proof of the equivalence $\mathrm{Sh}(T_G) \simeq G - \textbf{Sets}$.

Solution.

(1) Let $\{t_i\colon U_i \to U\}$ be a cover in T_G. It suffices to prove that

$$\coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i \to U$$

is a coequalizer diagram. Suppose V is a G-set and $f_i\colon U_i \to V$ is a collection of maps such that

$$\begin{tikzcd} U_i \cap U_j \arrow[r] \arrow[d] & U_i \arrow[d, "f_i"] \\ U_j \arrow[r, "f_j"] & V \end{tikzcd}$$

commutes for all i, j. Define $g\colon U \to V$ by $g(u) = g_i(u_i)$, where $t_i(u_i) = u$. This is well-defined on sets and equivariant by commutativity of the above diagram, and in fact it is the only definition compatible with all of the maps.

(2) Let $S = \bigsqcup_i G/H_i$ be a G-set and let $\mathcal{F} \in \mathrm{Sh}(T_G)$ be a sheaf. Then

$$\begin{aligned} \mathrm{Hom}_G(S, \mathcal{F}(G)) &= \mathrm{Hom}_G\left(\bigsqcup_i G/H_i, \mathcal{F}(G)\right) \\ &\cong \prod_i \mathrm{Hom}_G(G/H_i, \mathcal{F}(G)) \\ &\cong \prod_i \mathcal{F}(G)^{H_i} \\ &\cong \prod_i \mathcal{F}(G/H_i) \\ &\cong \mathcal{F}\left(\bigsqcup_i G/H_i\right) \\ &\cong \mathcal{F}(S). \end{aligned}$$

Thus $\mathcal{F} \cong h_{\mathcal{F}(G)}$.

The Yoneda embedding $T_G \to \mathrm{PSh}(T_G)$ has image in $\mathrm{Sh}(T_G)$ by (1), and is essentially surjective by what we just showed.

Remark 2. The previous exercise shows that T_G has the *canonical* topology, which is the finest topology for which every representable presheaf is a sheaf. If $\mathbf{C}$ is a Grothendieck topos, then the canonical coverings are those that are jointly epimorphic. In this case, the Yoneda embedding gives an isomorphism

$$\mathbf{C} \cong \mathrm{Sh}(\mathbf{C}).$$

The previous exercise is a special case of this upon taking $\mathbf{C} = G - \mathbf{Sets}$.

New exercise: Prove the result from the remark for the special case $\mathbf{C} = [\mathbf{D}, \mathbf{Sets}]$

Exercise 6. Let X be a scheme and $\mathcal{F}$ a sheaf on X_{zar}. Assume that $\mathcal{F}$ satisfies the sheaf axiom for single morphism fpqc coverings $V \to U$. Show that $\mathcal{F}$ is a sheaf on X_{fpqc}.

Solution. Let $\{t_i\colon U_i \to U\}$ be an fpqc cover, and consider the morphism $t\colon V = \bigsqcup_i U_i \to U$. We will show that t is an fpqc covering and that the sheaf axiom for the original family follows from that of t.

It's clear that t is flat since flatness is a local condition and t is locally one of the t_i's. Moreover, $t(U) = \bigcup_i t_i(U_i) = U$. Finally, let $W \subset U$ be an affine open. Then there exists affine opens $Z_{i_k} \subset U_{i_k} \subset V$ such that $W = \bigcup_k t_{i_k}(Z_{i_k}) = \bigcup_k t(Z_{i_k})$. So t is an fpqc cover.

Now assume that $\mathcal{F}\colon (\mathbf{Sch}/X)^{op} \to \mathbf{Sets}$ is a preasheaf for which

$$\mathcal{F}(U) \to \mathcal{F}(V) \rightrightarrows \mathcal{F}(V \times_U V)$$

is a pullback diagram. Using the facts that pullbacks commute with disjoint unions in the category of schemes [htt] and $\mathcal{F}$ turns coproducts into products,

$$\mathcal{F}(V \times_U V) \cong \mathcal{F}\left(\bigsqcup_{i,j} U_i \times_U U_j\right) \cong \prod_{i,j} \mathcal{F}(U_i \times_U U_j).$$

Therefore the original equalizer diagram is equivalent to the equalizer diagram

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j),$$

which is the sheaf condition for the original cover. Ideally, one should check that the maps in this equalizer diagram are the correct ones. ☐

Exercise 7. Assume that the representable functor h_Z is a sheaf on X_{fpqc} for all affine $Z \in \mathbf{Sch}/X$. Show that h_Z is a sheaf on X_{fpqc} for any $Z \in \mathbf{Sch}/X$.

solution...

Exercise 8. Show that $\mathcal{O}_X^{et}$ given by $\mathcal{O}_X^{et}(U) = \mathcal{O}_U(U)$ for $U \to X$ etale is an etale sheaf.

Solution. Consider the scheme $\mathbb{A}^1_X = X \times \mathbb{A}^1_{\mathbb{Z}}$, equipped with the projection $\mathbb{A}^1_X \to X$. Then for any $Z \in \mathbf{Sch}/X$,

$$(\mathbf{Sch}/X)\left(Z, \mathbb{A}^1_X\right) \cong \mathbf{Sch}\left(Z, \mathbb{A}^1_{\mathbb{Z}}\right) = \mathbf{Sch}\left(Z, \operatorname{Spec}\mathbb{Z}[x]\right) \cong \mathbf{Rings}\left(\mathbb{Z}[x], \mathcal{O}_Z(Z)\right) \cong \mathcal{O}_Z(Z).$$

Hence $\mathcal{O}_X^{et} \cong h_{\mathbb{A}^1_X}$. Since the etale site is subcanonical it follows that $\mathcal{O}_X^{et}$ is an etale sheaf.

References

[htt] Eric Wofsey (https://math.stackexchange.com/users/86856/eric-wofsey). *Do coproducts commute with pullback in the category of affine schemes?* Mathematics Stack Exchange. URL:https://math.stackexchange.co (version: 2017-06-12). eprint: `https://math.stackexchange.com/q/2320256`. URL: `https://math.stackexchange.com/q/2320256`.

[Vak] Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*.