ℓ-prime

X, Y - schemes

$f: X \to Y$

usually: everything fin type over a field. (say k)

Assume: $\ell \in \mathcal{O}_X^\times$.

Def: A local system on X is an étale sheaf, étale-locally trivial (const)

$(Loc_\ell(X) \xleftarrow{\ell} Loc_{\ell^2}(X) \xleftarrow{\ell} \ldots)$

lim

$\mathbb{Z}_\ell$-sh(X_{et})

Def: An étale sheaf $\mathcal{F}$ on X is constructible if $\exists$ $X_0 \subseteq \ldots \subseteq X_r$ closed emb. s.t. on $X_r \setminus X_{r-1}$ $\mathcal{F}$ is loc. const.

Note:
$(\mathcal{C},\tau)$-small
$\Rightarrow$ Ab.Sh$(\mathcal{C},\tau)$ has enough inj.

$\mathcal{M} = Sh(X_{\acute{e}t})$

$D(X)$: $\boxed{\text{bounded}}$ derived subcat. of $D(\mathcal{M})$ whose coho' obj. are constructible sheaves.

$D(X) \subset D(\mathcal{M})$

(1) $\mathcal{F}, \mathcal{G} \in D(X)$:

uses some kind of flat obj.

$\boxed{\mathcal{F} \otimes^L \mathcal{G}}$, $\boxed{R\mathcal{H}om(\mathcal{F},\mathcal{G})}$

$j_! \mathcal{O}_X$

$j : U \hookrightarrow X$ open

$\boxed{H^i(\mathcal{F}\otimes^L \mathcal{G}) =: \mathcal{T}or^i(\mathcal{F},\mathcal{G})} \in D(X)$

(1) $\mathcal{F} \boxtimes \mathcal{G} \in D(X \times_k X)$

(2) $L\Delta^*(\mathcal{F} \boxtimes \mathcal{G}) = \mathcal{F} \otimes^L \mathcal{G}$

$\Delta : X \hookrightarrow X \times X$

$$\begin{array}{c} X \\ f\downarrow \\ Y \end{array}$$

(1) $\underline{Rf_*}: D(X) \to D(Y)$

(2) $\underline{Lf^*}: D(Y) \to D(X)$

for:

(1) f proper, def. $f_! = f_*$

(2) if f is an open emb. $f_!$ is you know what. exact.

$$\boxed{\begin{array}{l} \underline{\text{Thm}}: \exists \text{ pseudofunctorial assignment } Sch/_k \to Cat \\ \qquad\qquad\qquad\qquad\qquad\qquad X \mapsto D(X) \\ X \xrightarrow{f} Y \text{ goes to} \\ (1)\ Rf_* \ \cancel{\text{if } f \text{ is proper or an open emb.}} \\ \qquad \text{if } f \text{ is proper} \\ (2)\ f_! \text{ if } f \text{ is open (ext. by 0)} \end{array}}$$

$\underline{\text{Rk}}$: (1) $\forall\ X \xrightarrow{f} Y$, $\exists\ X \xrightarrow{\text{open emb}} Z \xrightarrow{\text{proper}} Y$ (Nagata) $\sim$50s/60s

(2) $\underline{\text{cannot}}$ be made functorial

($\underline{\text{Ex}}$: prove this) $\underline{\text{hint}}$: $\dim X/Y \geq 2$.

Upshot ws $Rf_! : D(X) \to D(Y)$

"derived pf. w/ proper supp."

ex: for $X \xrightarrow{f} *$

Def:

- $H^n_c(X, \mathcal{F}) = H^n(Rf_! \mathcal{F})$
- $H^n(X, \mathcal{F}) = H^n(Rf_* \mathcal{F})$

Carbs:

$$\begin{tikzcd} X \arrow[r, hook, "j", "\text{open}"'] \arrow[dr, "f"'] & Y \arrow[d, "\pi", "\text{proper}"'] \\ & * \end{tikzcd}$$

$H^n_c(X, \mathcal{F}) := H^n(Y, j_! \mathcal{F}) :=$

$= H^n(R\pi_* \, j_! \mathcal{F}) =$

$= H^n(R\pi_! \cdot Rj_! \mathcal{F}) =$

$= H^n(R(\pi \circ j)_! \mathcal{F}) = H^n(Rf_! \mathcal{F})$.

Thm: In reasonable circumstances

$\exists$ right adjoint ~~to~~ RT!

call it $\pi^!$

strategy: triang.

Thm: $F : \mathcal{C} \to \mathcal{D}$ triang.

If (conditions)

$\Rightarrow \exists$ triang. adjoint

(Neeman).

e.g. $f : X \to Y$.

(separated) fin. type/k