Grothendieck topologies, sites, topoi

Let X be a top. space. $\mathrm{Open}(X)$ shall denote the category of open subsets of X with inclusions. Note that open covers $\{U_i \hookrightarrow U\}_{i \in I}$, $U \hookrightarrow X$ (i.e. a collection of opens s.t. $\bigcup_{i \in I} U_i \supseteq U$) satisfy

(i) $U \xrightarrow{=} U$ is a cover

(ii) If $\{V_{ij}\}_j$ covers U_i and $\{U_i\}_i$ covers U, then $\{V_{ij}\}_{i,j}$ covers U

(iii) Given a cover $\{U_i\}$ of U and an open $V \hookrightarrow U$, $\{V_i = U_i \cap V\}_i$ covers V

$U_i \cap V = U_i \times_U V$

need limits here

Grothendieck's insight: We can use this on any category $\mathcal{C}$ (that has finite limits): For each object U of $\mathcal{C}$ we have a distinguished set of families $\{U_i \to U\}_{i\in I}$ (coverings of U)

s.t.

(i) isomorphisms $V \to U$ are coverings

(ii) $V \to U$ a morphism and $\{U_i \to U\}_{i\in I}$ a covering
$\leadsto$ $U_i \times_U V$ exist and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering of V

(iii) $\{V_{ij} \to U_i\}_{j\in J}$ covering of U_i, $\forall i$, $\{U_i \to U\}_{i\in I}$ covering of U
$\leadsto$ $\{V_{ij} \to U_i \to U\}_{i,j\in I\times J}$ is a covering of U

This collection of coverings $Cov(\mathcal{C})$ is called a Grothendieck topology on $\mathcal{C}$.

$\mathcal{C}$ together with $\mathcal{J}$ is called a site.

Remark: • Open(X) (as seen above) has an obvious GT given by open covers

Exercise

- If X is a sober space (every irreducible closed subset is the closure of exactly one point), then we can recover X from the site described in the previous remark. (View $\mathrm{Open}(X)$ just as a poset)
- There are examples of spaces that can not be recovered from a GT
- There are GT that do not come from top. spaces

Examples:

- $\mathrm{Open}(X)$ with open covers
- $\mathcal{T}op$ with usual coverings
- $\mathcal{T}op$ with coverings $\{U_i \xrightarrow{f_i} U\}$ collectively surjective

$\mathcal{T}op_{ét}$

finite covering spaces of their image $U_i \to \mathrm{im}(f_i)$

- G a group we can define a site T_G on G-sets by taking coverings to be collectively surjective families $\{U_i \to U\}_{i \in I}$

- X a scheme :

  - X_{zar} is $\mathrm{Open}(X)$ as above — small Zariski site

  - X_{Zar} $Sch/_X$ with coverings $\{U_i \hookrightarrow U\}_i$ (over X) collectively surjective

    Big Zariski site

  - $X_{ét}$ is given by $Ét(X) \hookrightarrow (Sch/_X)_{ét}$ — full subcat of just étale morphisms $U \to X$

    and coverings $\{U_i \to U\}_i$ just collectively surjective

    necessarily étale by 2 out of 3

    small étale site

  - $X_{Ét}$ given by $Sch/_X$ and coverings $\{U_i \to X\}_i$ collectively surjective étale maps

    Big étale site

- $X_{nis} = X_{cd}$ take $\acute{E}t(X)$ with coverings (finite) surjective families of étale maps $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ s.t.

$\boxed{\forall u \in U\ \exists i \in I, u_i \in U_i \text{ above } u \text{ with}}$

Completely decomposed at u

$k(u) \xrightarrow{\cong} k(u_i)$ on residue fields

$$\begin{tikzcd} & U_i \arrow[d, "f"] \\ \mathrm{Spec}\, k(u) \arrow[r] \arrow[ur, dashed] & U \end{tikzcd}$$

small

Nisnevich site

$A \subset U$ affine open $\rightsquigarrow$

$A = \bigcup_{k} f_k(A_k)$ with $A_k \subset U_k$ affine open

$k \in$ finite

- X_{fpqc} take Sch/X with coverings quasi-compact (qc) families $\{U_i \longrightarrow U\}$ of flat (fp = fidèlement plat) morphisms fpqc site

We will call a faithfully flat morphism $V \xrightarrow{f} U$ fpqc if

- every quasi-compact open $W \subset U$ is the image of a quasi-compact open in V

Remark: faithfully flat and quasi compact morphisms are fpqc

Maps of sites: Let T_1, T_2 be sites. A functor

$$\mathrm{Cat}(T_2) \longrightarrow \mathrm{Cat}(T_1)$$

that preserves fibre products and transforms coverings into coverings is called a continuous map

$T_1 \longrightarrow T_2$.

Examples: • $Y \xrightarrow{f} X$ a map of top. spaces defines a cont. map of the corresponding sites iff f is continuous.

• The inclusions

$$X_{fpqc} \to X_{\acute{E}t} \to X_{\acute{e}t} \to X_{nis} \to X_{zar}$$

Sheaves on sites

A sheaf on a top. space X is a functor $Open(X)^{op} \to \mathcal{C}$ (valued in $\mathcal{C}$) satisfying the gluing and identity axiom.

Similarly, a sheaf on a site T is a functor $T \xrightarrow{\mathcal{F}} \mathcal{C}$ (= Sets or Ab...) s.t. all coverings $\{U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ give rise to an equalizer diagram

$$(s_i)_i \longmapsto (s_i|_{U_i\times_U U_j})$$

$$\mathcal{F}(U) \xrightarrow{\text{restriction}} \prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I}\mathcal{F}(U_i\times_U U_j)$$

$$(s_i)_i \longmapsto (s_j|_{U_i\times_U U_j})$$

Remark:

- $(\phi\to\phi)_\phi \rightsquigarrow \mathcal{F}(\phi) = \prod_\phi \cdots =$ final obj. in $\mathcal{C}$ ($*$ or 0 in Sets/Ab)

$U_i\cap U_j$ (fibred product over the terminal object $*$)

- Sheaves take disjoint unions to products (assume $\mathcal{C}$ has a terminal object $*$)

means $U_i\cap U_j = U_i\times_* U_j = \begin{cases}\phi, & i\neq j\\ U_i, & i=j\end{cases}$

$$U = \coprod U_i \rightsquigarrow \mathcal{F}(U) \cong \prod \mathcal{F}(U_i)$$

and implies

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j}\mathcal{F}(U_i\times_U U_j) = \prod_i \mathcal{F}(U_i)$$

- $U_i\times_U U_j = \phi$ for $i\neq j$

and $\cdot\ U_i\times_U U_i = U_i\times_{U_i} U_i = U_i$ and the projections are identical

equal

$$\rightsquigarrow \mathcal{F}(U) \cong \prod_i \mathcal{F}(U_i)$$

must be id by uniqueness

$$\begin{tikzcd}
U_1 \times U_2 \arrow[dr, "\exists!"] \arrow[drrr, bend left] \arrow[dddr, bend right] & & & \\
& \emptyset \arrow[dr, "\exists!"] \arrow[rr, bend left] \arrow[dd, bend right] & & \\
& & U_1 \times_U U_2 \arrow[r] \arrow[d] & U_2 \arrow[d] \arrow[dr] & \\
& U_1 \arrow[r] \arrow[drrr, bend right] & U & & \ast \\
& & & &
\end{tikzcd}$$

commutes b.c. $\emptyset$ is initial

commutes b.c. $*$ is terminal

and

$$\emptyset \to U_1 \times_U U_2 \to \emptyset \quad \text{(b.c. initial)}$$
$$\underbrace{\emptyset \to U_1 \times_U U_2 \to \emptyset}_{= id}$$

$\rightsquigarrow U_i \times_U U_j = \emptyset$ for $i \neq j$

- $Sh(T)$ = sheaves on the site T, $PSh(T)$ = presheaves on T = contravariant functors (to a fixed $\mathcal{C}$).
- A topos is a category equivalent to some $Sh(T)$ (with T a small site)

Example: Sheaves on (left) G-sets can be characterized as follows.

Note that we have an isomorphism

$g \cdot_{op} h := h \cdot g$

$$\begin{aligned} \mathrm{Aut}_G(G) &\cong G^{op} \\ (h \mapsto h \cdot g) &\leftarrow\!\shortmid g \end{aligned}$$

$\rightsquigarrow$ Any presheaf $\mathcal{F}$ on T_G leaves $\mathcal{F}(G)$ with a G-action via the above isomorphism

Claim: The category of G-sets is a topos:

Exercise: finish the proof

$$\begin{array}{ccc} \mathcal{F} & \longmapsto & \mathcal{F}(G) \\ Sh(T_G) & \underset{\longleftarrow}{\overset{\cong}{\longrightarrow}} & G\text{-sets} \\ h_{..} & \longleftarrow\!\shortmid & X \end{array}$$

$$\text{Hom}_G(-,X)$$

Exercise: Show y_X is a sheaf

Clues:

1.) $$\mathcal{F}(S) = \prod_{O \in S/G} \mathcal{F}(O)$$

by remark under def. of sheaves above

Any G-set S decomposes into its orbits

$$S = \coprod_{O \in S/G} O.$$

Fibred products in G-sets are just fibred products in sets

2.) $$\mathcal{F}(G/H) = \mathcal{F}(G)^H$$ for $H \subset G$ a subgroup

elements fixed by H

The sheaf axiom on the covering $\{G \to G/H\}$ says

$$s \longmapsto (s)_{h \in H}$$

$$\mathcal{F}(G/H) \to \mathcal{F}(G) \rightrightarrows \mathcal{F}(G \times_{G/H} G)$$

$$G \times_{G/H} G \cong G \times H = \coprod_{h \in H} G, \qquad (g_1, g_2) \mapsto (g_1, g_1 g_2^{-1})$$

$s \mapsto (hs)_{h \in H}$

$\leadsto \rightrightarrows \prod_{h \in H} \mathcal{F}(G)$

These equalize on precisely the fixed points of H

$\leadsto \mathcal{F}(G/H) \cong \mathcal{F}(G)^H$

$\leadsto$ $\mathcal{F}$ is determined by global sections

Remark: • In this case all representable functors h_X are sheafs. Such a site is called sub canonical, or a site less fine than the canonical one

• The canonical topology on a category $\mathcal{C}$ is the finest topology s.t. all h_X are sheaves (of sets). It is given by taking coverings to be all families of universal effective epimorphisms i.e.

- What we have just shown proves that $\mathcal{T}_{can}$ is the canonical topology on G-Sets.

X_{fpqc} is subcanonical

Proposition Let $X_{fpqc} \xrightarrow{\mathcal{F}} Set$ be a presheaf. $\mathcal{F}$ is a sheaf iff

(i)
$$\begin{tikzcd} X_{fpqc} \arrow[rr, "\mathcal{F}"] \arrow[dr] & & Set \\ & X_{Zar} \arrow[ur, "\mathcal{F}|_{Zar}"'] & \end{tikzcd}$$
the restriction to the big Zariski site is a sheaf

(ii) $\mathcal{F}$ satisfies the sheaf axioms for single faithfully flat coverings $V \longrightarrow U$ for V, U affine

Exercise: Reduce more general coverings to single morphism coverings under assumption of (ii).

proof: Take an fpqc morphism $V \xrightarrow{f} U$ (we can reduce to these single morphism coverings). We pick an open cover $\{V_i \hookrightarrow V\}$ of quasi-compact opens s.t. their images $f(V_i) =: U_i \hookrightarrow U$ are open and affine.
Then cover the V_i with finitely many affine opens V_{ij}.

finite cover of faithfully flat maps between affines
$\{V_{ij} \to U_i\}_j$

$\leadsto \mathcal{F}(U_i) \to \prod_{a,b} \mathcal{F}(V_{ia} \times_{U_i} V_{ib})$ is injective by (ii)

In particular, each factor $\mathcal{F}(U_i) \hookrightarrow \prod_j \mathcal{F}(V_{ij})$ is injective

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d, hook] & \mathcal{F}(V) \arrow[d, hook] \\
\prod_i \mathcal{F}(U_i) \arrow[r, hook] & \prod_{ij} \mathcal{F}(V_{ij})
\end{tikzcd}$$

injective b.c. $\mathcal{F}$ is a Z. sheaf

$\leadsto$ top map is injective (identity axiom)

Sketch for the gluing axiom:

We have a commutative diagram

equalizers b.c. $\mathcal{F}$ is a $\mathbb{Z}$-sheaf

$$
\begin{tikzcd}
\mathcal{F}(U) \arrow[r] \arrow[d] & \mathcal{F}(V) \arrow[r, shift left, "pr_1^*"] \arrow[r, shift right, "pr_2^*"'] \arrow[d] & \mathcal{F}(V \times_U V) \arrow[d] \\
\prod_i \mathcal{F}(U_i) \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \mathcal{F}(V_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \prod_i \mathcal{F}(V_i \times_{U_i} V_i) \\
\prod_{i,j} \mathcal{F}(U_i \cap U_j) \arrow[r] & \prod_{i,j} \mathcal{F}(V_i \cap V_j) &
\end{tikzcd}
$$

equalizer

(need to show that $V \to U$ with U affine gives equalizer and products of equalizers are equalizers (limits commute with limits))

We omit this. This is where the work is. See "FAG explained"

injective b.c. we have shown the identity axiom

To see that the top row is an equalizer it suffices to show that any element $s \in \mathcal{F}(V)$ s.t. $pr_1^*(s) = pr_2^*(s)$ comes from some $s' \in \mathcal{F}(U)$. We move such an s to the orange row. By commutativity it equalizes the restriction maps there too. Hence, comes from some $(s_i)_i \in \prod_i \mathcal{F}(U_i)$. Again by comm. and the injectivity of all single horizontal arrows, $(s_i)_i$ equalizes the restriction maps of the 1st column.

· It equalizes the restriction maps of the 1st column.

$\rightsquigarrow (s_i)_i$ glues to some $s' \in \mathcal{F}(U)$ by the 1^{st} column equalizer.

By comm. s' maps to s. $\square$

Remark: The fpqc site was the finest site that we have introduced for schemes.

The above statement holds for all coarser sites (Et, ét, Zar, big Zar, cd, ...)

(You may replace big Zar with small Zar and faithfully flat with étale, cd, ...).

Proposition: The fpqc site is subcanonical.

proof: Let $Z \in Sch/X$. We have to show that $h_Z = Hom_X(-, Z)$ is an fpqc sheaf. We check our condition. First, it is easy to see that h_Z is a Z-sheaf. ✓ Thus, it suffices to show that for a faithfully flat map of rings $A \to B$

$$h_Z(\mathrm{Spec}\,A) \to h_Z(\mathrm{Spec}\,B) \rightrightarrows h_Z(\mathrm{Spec}\,B\otimes_A B)$$

is an eq. diagram. For $Z = \mathrm{Spec}\,C$ affine this becomes

$$\mathrm{Hom}_A(C,A) \to \mathrm{Hom}_A(C,B) \rightrightarrows \mathrm{Hom}_A(C,B\otimes_A B)$$

Lemma:

$$0 \to A \xrightarrow{f} B \xrightarrow{g} B\otimes_A B \quad \text{is exact}$$
$$b \mapsto 1\otimes b - b\otimes 1$$

This implies the above claim.

proof of Lemma: 1^{st}: any faithfully flat map $A \to B$ is injective (look at the SES $0 \to \ker f \to A \to A/\ker f \to 0$) ✓

2^{nd}: Suffices to show that $A \xrightarrow{f} B$ has a section s ($s\circ f = \mathrm{id}_A$).

NTS $\ker g = f(A) \cong A$. "$\supseteq$" clear. "$\subseteq$": given s let $1\otimes b - b\otimes 1 = 0$

$1\otimes b = b\otimes 1$ apply $B\otimes_A B \to B$, $x\otimes y \mapsto x\cdot f(s(y))$

$\rightsquigarrow \quad f(s(b)) = b \in f(A)$ ✓

3rd: In general we don't have a section, but sometimes we do and we can base change until we do.

$B \to B \otimes_A B, \; b \mapsto b \otimes 1$ has a section (multiplication) and we get the sequence of this map by tensoring our sequence $- \otimes_A B$ ($A \to B$ f.f. $\rightsquigarrow$ exact as $- \otimes_A B$ is) $\square$

If Z is not affine we can still cover it with affines and reduce to those (Exercise). $\square$

This again implies that all coarser sites we have seen (in particular Ét and Et) are subcanonical.

Remark: The exactness of the sequence in the lemma extends to exactness of a LES

$$0 \to A \to B \to B^{\otimes 2} \to B^{\otimes 3} \to \cdots$$ Amitsur complex

using the same trick due to Amitsur.

Legend has it that Grothendieck was blown away by this trick. I could not find a reference for this.

Examples: • $Z \in \text{Ét}(X) \rightsquigarrow h_Z$ is a sheaf

Important: • $Z = X \times \operatorname{Spec} \mathbb{Z}[t]/(t^n - 1) =: \mu_n$
n-th roots of unity

• $Z = \mathbb{A}^1_X =: \mathbb{G}_a$

• $Z = X \times \operatorname{Spec} \mathbb{Z}[t^{\pm 1}] = \mathbb{G}_m$ sheaf of units

• $\mathcal{O}_X^{\text{ét}}$ given by $\mathcal{O}_X^{\text{ét}}(U) = \mathcal{O}_U(U)$ for $U \to X$ étale

structure sheaf on $X_{\text{ét}}$

Structure sheaf on X_{et}

- locally constant sheaves

Exercise: Show that the above are sheaves.

Descent theory (short excursion)

theory of producing something global out of local data

Examples: • Zariski descent: We can glue morphisms/schemes/sheaves of a compatible datum of these on opens

- fpqc descent for quasi-coh. sheaves

fpqc

... sheaves relative to a covering $\{U_i \to U\}$

A descent datum for qc-sheaves relative to a covering ...

is • a qcs $\mathcal{F}_i$ on U_i $\forall i$

• isos $\mathcal{F}_i|_{U_{ij}} \xrightarrow{\cong} \mathcal{F}_j|_{U_{ij}}$ satisfying the cocycle condition

A descent datum descends or is effective if it glues to a qcs only

Thm (Grothendieck): Every descent datum of qcs rel. to an fpqc covering is effective.

Remark: In general we do not have fpqc or even étale descent for schemes.

There is a lot more to say here. Stephan will talk about this later.