Abelian sheaves

$\mathcal{C}$ small site
$PAb(\mathcal{C})$ abelian presheaves
$Ab(\mathcal{C})$ abelian sheaves

<u>Fact</u>: The inclusion $Ab(\mathcal{C}) \hookrightarrow PAb(\mathcal{C})$ has a left adjoint that commutes with finite limits.

<u>Note</u>: Smallness of $\mathcal{C}$ is important here.

<u>Consequences</u>:

1. $Ab(\mathcal{C})$ is an abelian category.
2. [illegible]

2. $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$ exact in $Ab(\mathcal{C})$

$\iff \forall U \in \mathcal{C}$, $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$ is exact.

3. $\mathcal{G} \to \mathcal{H} \to 0$ exact in $Ab(\mathcal{C})$

$\iff \forall U \in \mathcal{C}$, $s \in \mathcal{H}(U)$ $\exists$ cover $\{U_i \to U\}$ s.t. $s|_{U_i} \in \mathrm{im}(\mathcal{G}(U_i) \to \mathcal{H}(U_i))$.

<u>Fact</u>: $Ab(\mathcal{C})$ has enough injectives.

Hence if $G : Ab(\mathcal{C}) \to Ab$ is left exact, we can compute the right derived functors:

1. Take an injective resolution of some $\mathcal{F} \in Ab(\mathcal{C})$:

$$0 \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$$

st each I^k is injective, and

$$H^k(I^\bullet) = \begin{cases} \mathcal{F} & \text{if } k=0 \\ 0 & \text{otherwise.} \end{cases}$$

2. Apply G:

$$0 \to GI^0 \to GI^1 \to \cdots$$

3. Take cohomology: $R^iG(\mathcal{F}) = H^i(GI^\bullet)$.

Fact: Can replace "injective" in 1 by G-acyclic.

($\mathcal{F}$ is G-acyclic if $R^iG(\mathcal{F}) = 0$ for $i > 0$.)

Sheaf cohomology

Sheaf cohomology

Let $X \in \mathcal{C}$, consider $\Gamma_X : Ab(\mathcal{C}) \longrightarrow Ab$
$$\tilde{F} \longmapsto \tilde{F}(X).$$

This is left exact.

Define: $H^i(X;\tilde{F}) = R^i\Gamma_X(\tilde{F})$.

If X is a terminal object in $\mathcal{C}$, write $H^i(\mathcal{C};\tilde{F}) = H^i(X;\tilde{F})$.

<u>Example</u>: Let X be a locally contractible, paracompact topological space.

Then $H^q_{sing}(X;\mathbb{Z}) \cong H^q(X;\underline{\mathbb{Z}})$.

<u>Proof sketch</u>:

Presheaf on $\mathrm{Open}(X)$: $U \mapsto C^q_{sing}(U)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \shortparallel$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ q-th singular cochains in U

Sheaf on $\mathrm{Open}(X)$: C^q = sheafification of C^q_{sing}

If U is contractible,

$$0 \to \mathbb{Z} \to C^0_{sing}(U) \to C^1_{sing}(U) \to \cdots$$

is exact. Since X is locally contractible,

$$0 \to \mathbb{Z} \to C^0_{sing} \to C^1_{sing} \to \cdots$$

is locally exact. Therefore

$$0 \to \underline{\mathbb{Z}} \to C^0 \to C^1 \to \cdots$$

is an exact sequence of sheaves.

<u>Exercise</u>: Show that

$$\boxed{0 \to K^\bullet \to C^\bullet_{sing} \to C^\bullet \to 0}$$

is an exact sequence in $PAb(X)$,

where $K'(U) = \left\{ \varphi \in C^q_{sing}(U) \;\middle|\; \begin{array}{l} \text{for some open cover } \\ \{U_i \hookrightarrow U\},\ \varphi|_{U_i} = 0 \\ \text{for all } i \end{array} \right\}$

Hint: Use description of $C^\bullet$ as compatible families of germs and paracompactness.

A sheaf $\hat{F}$ on $\mathrm{Open}(X)$ is called flasque if $\hat{F}(X) \twoheadrightarrow \hat{F}(U)$ is surjective for all $U \in \mathrm{Open}(X)$.

Fact: Flasque sheaves are acyclic.

Exercise 2: C^q is flasque.

Therefore $H^*(C^\bullet(X)) = H^*(X;\mathbb{Z})$.

<u>Claim</u>: $H^*(K^\bullet(X)) = 0$

Now SES from Ex 1

$$0 \to K^\bullet(X) \to C^\bullet_{sing}(X) \to C^\bullet(X) \to 0$$

yields LES in cohomology:

$$\begin{array}{ccc} H^*(C^\bullet_{sing}(X)) & \cong & H^*(C^\bullet(X)) \\ \wr\wr & & \| \\ H^*_{sing}(X) & & H^*(X;\mathbb{Z}) \end{array}$$

Proof of claim: Let $\mathcal{U} = \{U_i \hookrightarrow X\}$ be an open cover.

$$C_\bullet^{\mathcal{U}}(X) = \text{simplices in } X \text{ contained in some } U_i$$
$$= \varinjlim C_\bullet(U_i)$$

Theorem: $C_\bullet^{\mathcal{U}}(X) \hookrightarrow C_\bullet(X)$ is a chain htpy equivalence

Let $C^{\bullet,\mathcal{U}}_{sing}(X) = \mathrm{Hom}(C_\bullet^{\mathcal{U}}(X), \mathbb{Z})$.

Have SES

$$0 \to K^{\bullet,\mathcal{U}}(X) \to C^\bullet(X) \to C^{\bullet,\mathcal{U}}(X) \to$$

$$0 \to K(X) \to C_{sing}(X) \to C_{sing}(X) \to 0$$

$\quad \Vert \qquad\qquad\qquad\qquad$ q-iso by thm.

$$\{ \varphi \in C^{\bullet}_{sing}(X) \mid \varphi|_{U_i} = 0 \text{ for all } i \}$$

LES in cohomology $\Rightarrow$ $H^{*}(K^{\bullet,\mathcal{U}}(X)) = 0$

<u>Note</u>: $\varinjlim_{\mathcal{U}} K^{\bullet,\mathcal{U}}(X) = K^{\bullet}$

<u>Recall</u>: 1. taking filtered colimits of abelian groups is exact

2. cohomology commutes with exact functors.

Hence: $H^*(K^\circ(X)) = H^*(\varinjlim_{\mathcal{U}} K^{\bullet,\mathcal{U}}(X))$

$\cong \varinjlim H^*(K^{\bullet,\mathcal{U}}(X)) = 0.$ $\square$

Example: X manifold, then $H^*(X;\mathbb{R}) \cong H^*_{dR}(X)$.

Note that these examples show deRham's theorem: $H^*_{dR}(X) \cong H^*_{sing}(X;\mathbb{R})$ (using argument from the first example with $\mathbb{R}$ insted of $\mathbb{Z}$ coefficients)