We will now define the étale homotopy type of a scheme and hopefully see some nice properties of it reminiscent of classical htpy. theory. We will realize the ét. htpy type as the shape of it's associated Étale topos.

References:

- Appendix of HA
- "Higher Galois theory" Hoyois
- "The étale Symmetric Künneth theorem" Hoyois

Classically:

cw complexes or kan complexes $=$

$$\begin{array}{ccc} \mathcal{S} & & \mathcal{T}op \\ \shortparallel & & \shortparallel \\ \mathrm{Spaces} & \rightsquigarrow & \mathrm{Topoi} \\ & & \\ X & & \mathcal{X} = \mathrm{Shv}(X) \end{array}$$

The global sections functor

$$\mathcal{X} \xrightarrow{\pi_*} \mathrm{Shv}(*) \simeq \mathcal{S}$$

$$\downarrow \pi \qquad\qquad \pi_* \uparrow \ \downarrow \pi^*$$

$$* \qquad\qquad \mathcal{S} = \mathrm{Shv}(*)$$

is right adjoint to the inverse image functor

$$\mathcal{X} \xleftarrow{\pi^*} \mathrm{Shv}(*) \simeq \mathcal{S}$$

The composition

$$\Pi_\infty \mathcal{X} : \ \mathcal{S} \xrightarrow{\pi^*} \mathcal{X} \xrightarrow{\pi_*} \mathcal{S}$$

is called the shape of $\mathcal{X}$.

we have $\pi_* \pi^* (T) = \Gamma(\pi^*(T)) \simeq \mathrm{Map}(X, T)$ ← computes cohomology with const. coeff. of X via $T = K(n,A)$

$\pi^*(T)$ = const. sheaf T on X

controls the weak htpy. type of X

⇝ the shape of $\mathcal{X}$ "is" the weak htpy. type of X

This construction very suitable for generalizations to more general topoi.

Shapes

Let $\mathcal{X}$ be an ∞-topos.

an ∞-cat that is
- presentable
- colimits are stable under pullback
- coproducts are disjoint
- groupoid objects can be delooped

A geometric morphism between topoi $\mathcal{Y} \xrightarrow{f} \mathcal{X}$ is a functor $\mathcal{X} \xrightarrow{f^*} \mathcal{Y}$ preserving finite limits and all colimits.

Equivalently, it is an adjoint pair (f^*, f_*) of functors s.t. f^* preserves finite limits

limits.

Since f^* preserves finite limits it admits a pro-left adjoint

$$\mathcal{Y} \xrightarrow{f_!} \mathrm{Pro}(\mathcal{X})$$

formal cofiltered limits = functors $\mathcal{J} \to \mathcal{X}$ s.t. $\mathcal{J}$ is small and cofiltered

It is given by

$$f_!(Y)(X) \simeq \mathrm{Map}_{\mathcal{Y}}(Y, f^*(X)).$$

The functor corepresented by the final object of $\mathcal{X}$ will be denoted by

$\mathcal{X} \xrightarrow{\pi_*} \mathcal{S}$ and a left adjoint by π^*. This is a geometric morphism to $\mathcal{S} \simeq \mathrm{Shv}(*)$ (it is actually unique).

constant sheaf functor

$\mathcal{S} \to \mathcal{X}$

The shape of $\mathcal{X}$ is given by

final object in $\mathcal{X}$

$$\Pi_\infty \mathcal{X} := \pi_! 1 \in \mathrm{Pro}(\mathcal{S})$$

which is given by $\pi_* \pi^*$ as a left exact functor $\mathcal{S} \to \mathcal{S}$

sends a space to the global sections of the associated constant sheaf

This is the shape functor $\mathcal{T}op \xrightarrow{\Pi_\infty} \mathrm{Pro}(\mathcal{S})$,

where $\mathcal{T}op$ is the ∞-category of ∞-topoi and geometric morphisms.

Homotopy types of schemes

Let X be a scheme and τ a pretopology on the category of schemes.

↑ a basis for a Grothendieck topology:

- isos. cover
- covering families stable under pullback
- transitivity (see 2nd talk)

We consider the small τ-site of X (full subcat. of Sch_X gen. spanned by members of τ-coverings of X with τ-topology).

We will denote the corresponding ∞-topos of sheaves of spaces on this site by X_τ.

The τ-homotopy type of a scheme X is the shape of the ∞-topos X_τ

$$\Pi_\infty^\tau X := \Pi_\infty X_\tau .$$

This gives us a functor $\mathrm{Sh} \xrightarrow{\Pi_\infty^\tau} \mathrm{Pro}(\mathcal{S})$.

Remember, we have

$$\mathrm{Map}(1, \pi^* K) \simeq \mathrm{Map}(\Pi_\infty \mathfrak{X}, K) \quad \text{for } K \in \mathcal{S}.$$

τ-cohomology in constant sheaf $\pi^* K$ of X

abelian/continuous cohomology of $\Pi_\infty \mathfrak{X}$ in an abelian group A if we pick K to be $K(A, n)$

$\leadsto$ we can compute sheaf τ-cohomology via the τ-homotopy type.

$\mapsto$ Let's explain this in a little more detail.

Let $\mathcal{T}$ be a classical topos and $\mathcal{F} \in Ab(\mathcal{T})$ an abelian sheaf.

As before we have a geometric morphism

$$\begin{tikzcd} \mathcal{T} \arrow[r, shift right, "\pi_*"'] & Set \arrow[l, shift right, "\pi^*"'] \end{tikzcd}$$

which I will denote by

$$\begin{tikzcd} \mathcal{T} \arrow[r, shift right, "\Gamma_*"'] & Set \arrow[l, shift right, "\Gamma^*"'] \end{tikzcd}$$

taking global sections

and this restricts to abelian objects

$$\begin{tikzcd} Ab(\mathcal{T}) \arrow[r, shift right, "\Gamma_*^{Ab}"'] & Ab(Set) \arrow[l, shift right, "\Gamma^*_{Ab}"'] \end{tikzcd}.$$

We have defined cohomology of $\mathcal{J}$ with coefficients in $\mathcal{F}$ by

$$H^n(\mathcal{J}, \mathcal{F}) := H_{-n}(\mathbb{R}\Gamma_*^{Ab}\mathcal{F}).$$

Eilenberg - MacLane objects

We would like objects representing cohomology in degree n for any abelian sheaf $\mathcal{F}$. We can find those using Dold-Kan.

Note that $\mathcal{J} \xrightarrow{\Gamma_*} \mathrm{Set}$ is representable by the final object $1 =: * \in \mathcal{J}$:

$$\Gamma_*(\mathcal{F}) = \mathrm{Hom}(*, \Gamma_*(\mathcal{F})) = \mathrm{Hom}(\Gamma^*(*), \mathcal{F})$$
$$= \mathrm{Hom}(*, \mathcal{F})$$

Let $\mathbb{Z}_{\mathfrak{J}} := \Gamma^*_{Ab}(\mathbb{Z})$. Then

$$\mathbb{R}\Gamma^{Ab}_*(\mathcal{F}) = \mathbb{R}\mathrm{Hom}(\mathbb{Z}, \Gamma^{Ab}_*(\mathcal{F})) = \mathbb{R}\mathrm{Hom}(\mathbb{Z}_{\mathfrak{T}}, \mathcal{F})$$

and cohomology becomes

viewed as chains concentrated at 0

$$H^n(\mathfrak{J}, \mathcal{F}) = H_0(\mathbb{R}\Gamma_* \mathcal{F}[n]) = H_0(\mathbb{R}\mathrm{Hom}(\mathbb{Z}_{\mathfrak{T}}, \mathcal{F})[n]) = [\mathbb{Z}_{\mathfrak{T}}, \mathcal{F}[n]]_{h_0(Ch(Ab(\mathfrak{T})))}$$

$$= [K(\mathbb{Z}_{\mathfrak{T}}), K(\mathcal{F}[n])]_{sAb(\mathfrak{T})} = [K(\Gamma^*(\mathbb{Z})), K(\mathcal{F}[n])]_{sAb(\mathfrak{T})}$$

Dold-Kan

$sA \underset{N}{\overset{K}{\rightleftarrows}} C_{\geq 0}(A)$

is an equivalence of categories

const sgroup on $\mathbb{Z}$

$$= [\Gamma^*(K(\mathbb{Z})), K(\mathcal{F}[n])]_{sAb(\mathfrak{T})}$$

$=: \mathbb{Z}_\bullet$ const at $\mathbb{Z}_{\mathfrak{T}}$

$= [\mathbb{Z}_\bullet, K(\mathcal{F}[n])]_{sAb(\mathcal{T})}$

K both left and right adjoint to N. $\Leftrightarrow$ K commutes with Γ^* $\Leftrightarrow$ $N \circ \Gamma_* \cong \Gamma_* \circ N$

$\Uparrow$ true b.c. N is defined via kernels and Γ_* is left exact thus respects them

$\parallel$

$\mathbb{Z} \otimes *$, where $* \in \mathcal{T}$ and we have an adjunction $sAb(\mathcal{T}) \underset{U}{\overset{\mathbb{Z}\otimes -}{\rightleftarrows}} s\mathcal{T}$

$$= [*, UK(\mathcal{F}[n])]_{s\mathcal{T}}$$

Eilenberg-MacLane object for $\mathcal{F}$ in degree n

$=: K(\mathcal{F}, n)$

Example:

$\mathcal{T} = Set \rightsquigarrow Ab(Set) = Ab$

$$\pi_k(K(\mathcal{F}, n)) = H_k(\mathcal{F}[n]) = \begin{cases} \mathcal{F}, & k = n \\ 0, & \text{else} \end{cases}$$

With the above we can just take cohomology in degree n with coeff. in $\mathcal{F}$ to be

$$\boxed{\pi_0(\mathbb{R}\Gamma_*(K(\mathcal{F},n)))}.$$

In our original notation $(\pi = \Gamma, \mathbb{1} = *)$ we had

constant sheaf of abelian groups

$$\mathrm{Map}(\mathbb{1}, \pi^* K(A,n)) \simeq \mathrm{Map}(\pi^\infty \mathcal{X}, K(A,n))$$

$$\pi^* K(A,n) = K(\pi^* A, n)$$

π_0 of this is as we have seen given by cohomology of the topos in degree n

reference for model cat version: "Étale homotopy theory" lecture with coefficients in A

notes by Tomer Schlank

Remark

Note that the above was appealing to the category $s\mathcal{I}$ and I sweeped under the rug the relationship between $s\mathcal{I}$ (with a certain model structure) and corresponding ∞-topoi.

Given a Grothendieck topos $\mathcal{I}$, there is a model structure on $s\mathcal{I}$ (Joyal - Jardine). There are also two ∞-topoi

$$\Delta\mathcal{I} = \text{stacks on } \mathcal{I}$$

$$(\Delta\mathcal{I})^{hyp} = \text{stacks on } \mathcal{I} \text{ satisfying hyperdescent}$$

associated to τ. The latter one is equivalent to the theory of Joyal-Jardine. It is a hypercomplete theory (Whitehead's theorem holds).

One gets the hypercompletion by inverting ∞-connected morphisms, i.e. morphisms that are n-connected for all n.

Even if $\Delta\tau$ is not hypercomplete it has better formal properties than $(\Delta\tau)^{hyp}$ (in cases where these are not the same).

$X_{\acute{e}t}$ is rarely hypercomplete

X_{Nis} is hypercomplete for X Noetherian of finite dimension (Morel - Voevodsky)

The discussion about Eilenberg-MacLane objects can be carried out in $\Delta\tau$ (in fact, cohomology with coefficients in an Eilenberg-MacLane object agree in $\Delta\tau$ and $(\Delta\tau)^{hyp}$).

Reference: "On ∞-topoi" by Jacob Lurie