étale homotopy type of $\operatorname{Spec}(\mathbb{Z})$

$=: X$

This has been outlined by Tomer Schlank in a mathoverflow post

Last time we defined the shape

$\pi_\infty^{ét}(X) \in \mathrm{Pro}(\mathcal{S})$ of a scheme X.

We remark that $\mathrm{Pro}(\mathcal{S})$ can be naturally organized in an ∞-cat. Such that we have analogues of the Hurewicz and the universal coefficients theorem (see "étale homotopy" by Artin-Mazur for Hurewicz)

we will in good faith believe this works

Want to compute $\pi_*(\pi_\infty^{ét} X)$

Strategy: 1.) Compute $H^n(X, \mathbb{Z}/n) = 0$ except a $n=0$

2.) Profiniteness:

[illegible] $\pi_*(\Pi_\infty X)$ are profinite

3.) Harewicz $\Pi_n = H^n$

$\uparrow$ profinite

$\uparrow$ by universal coeff this must be 0 b.c. $H^n(-,\mathbb{Z}/n) = 0$

$\pi_1(X) = 0$ $\nearrow$

X is simply connected

Finite cohomology of X

We know (from Eivind) the cohomology groups

$H^r(X, \mathbb{G}_m)$:

r	0	1	2	3	$2r \geq 4$	$2r+1 \geq 5$
H^r	$\mathbb{Z}/2$	0	0	$\mathbb{Q}/\mathbb{Z}$	$\mathbb{Z}/2$	0

Proposition

$$H^r_c(X, \mathbb{G}_m) = \begin{cases} \mathbb{Q}/\mathbb{Z} & , r = 3 \\ 0 & , \text{else} \end{cases}$$

proof We have a LES

$$\cdots \longrightarrow H^r_c(X, \mathbb{G}_m) \longrightarrow \underset{\underset{0 \text{ for } r \neq 0,3,\, 2r \geq 4}{\uparrow}}{H^r(X, \mathbb{G}_m)} \longrightarrow \bigoplus_{v \notin X} H^r(K_v, \mathbb{G}_{m_v}) \longrightarrow H^{r+1}_c(X, \mathbb{G}_m) \longrightarrow \cdots$$

$r=2$

$\longrightarrow$

$$0 \longrightarrow H^2_c \longrightarrow \overset{0}{\cancel{H^2}} \longrightarrow \bigoplus_{v \notin X} \underset{\mathbb{Z}/2}{\underset{\parallel}{Br(K_v)}} \longrightarrow H^3_c \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

$\leadsto$ $H^2_c = 0$

$H^3_c \cong \mathbb{Q}/\mathbb{Z}$

proof omitted

$2r \geq 4$

$\longrightarrow$

$$0 \longrightarrow H^{2r}_c \longrightarrow \mathbb{Z}/2 \xrightarrow[\text{3.3 in Eivind's notes}]{\cong} \mathbb{Z}/2 \longrightarrow H^{2r+1}_c \longrightarrow 0$$

$\leadsto$ $H^r_c = 0$ for $r \geq 4$

" no global sections with compact support "

$\Downarrow$

$r=0$

$$0 \to \underset{\underset{0}{\parallel}}{H^0_c} \to \underset{\underset{\mathbb{Z}/2}{\parallel}}{H^0} \xrightarrow{=} \underbrace{\bigoplus_{v\notin X} H^0(K_v, \mathbb{G}_{m,v})}_{\mathbb{Z}/2} \to H^1_c \to 0$$

$$\to H^1_c = 0$$

□

Corollary

$$H^r_c(X, \mu_n) = \begin{cases} \mathbb{Z}/n, & r = 3 \\ 0, & \text{else} \end{cases}$$

proof Use SES of sheaves

$$\mu_n \to \mathbb{G}_m \to \mathbb{G}_m$$

SLES

$$0 \to H^3_c(X,\mu_n) \to \mathbb{Q}/\mathbb{Z} \xrightarrow{\times n} \mathbb{Q}/\mathbb{Z} \to H^4_c(X,\mu_n) \to 0$$

$$H^3_c = \ker = \mathbb{Z}/n \qquad \text{surj.} \to H^4_c(X,\mu_n) = 0$$

$\square$

We are interested in the non-compact groups with finite coeff.
To get those we use

Theorem (Artin-Verdier duality)

We have a non-degenerate pairing

$$H^r(X, \mathbb{Z}/n) \times H_c^{3-r}(X, \mu_n) \longrightarrow H_c^3(X, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$$

This implies

$$H^r(X, \mathbb{Z}/n) \cong \begin{cases} \mathbb{Z}/n, & r = 0 \\ 0, & \text{else} \end{cases} \quad \forall n$$

homotopy groups of Spec(ℤ)

We will use Hurewicz and the fact that $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Z})) = 0$.

$\uparrow$

Y (n-1)-connected, then

$$\pi_n(Y,y) \xrightarrow{\simeq} H^n(Y,\mathbb{Z})$$

for $n \geq 1$

$\|$ ← omitted

$\pi_1(\pi_\infty^{\acute{e}t}(X))$

$\Uparrow$ X has no unramified étale covers (follows by a thm. of Minkowski).

Theorem (profiniteness)

étale cover is irred. → ⟸ is connected

(X normal implies this)

Let X be a connected Noetherian scheme that is geometrically unibranch. Then $\pi_\infty^{\acute{e}t}(X)$ has profinite homotopy groups.

proof: references: • "étale homotopy", Mazur, Artin

• "étale homotopy type" notes by Tomer Schlank

Proposition $\pi_*(\mathrm{Spec}(\mathbb{Z})) = 0$

proof: Let $S := \pi_\infty^{\text{ét}}(\mathrm{Spec}\,\mathbb{Z})$. As mentioned before we have $\pi_1(S) = 0$.

$$\leadsto \pi_2(S) \overset{\text{Hurewicz}}{\cong} H^2(S)$$

profinite

but $H^2(X, \mathbb{Z}/n) = 0$ $\forall n \geq 2$

$$\xrightarrow[\text{Coeff.}]{\text{univ.}} \quad H^2(X,\mathbb{Z}) = 0$$

$$\leadsto \quad \pi_3(S) \cong H^3(X,\mathbb{Z}) \quad \ldots$$

$$\text{induction} \quad \pi_*(S) = 0 \qquad \square$$

Remark: This does not show that $\pi_\infty^{ét}\,\mathrm{Spec}(\mathbb{Z}) \simeq *$!

Postnikov towers in the étale topos do not necessarily converge. As mentioned in the last lecture, $X_{ét}$ is also rarely hypercomplete, so the Whitehead theorem does not hold.

What we have shown is that the protruncated étale shape

$$\Pi_{<\infty}^{ét}(\mathrm{Spec}(\mathbb{Z})) := \underset{\substack{\uparrow \\ \text{protruncation}}}{\tau_{<\infty}}(\Pi^{et}(\mathrm{Spec}(\mathbb{Z})))$$

is trivial.

Let $\mathcal{S}_{<\infty} \hookrightarrow \mathcal{S}$ be the full subcat. of truncated objects. The inclusion

$$\begin{tikzcd} \mathrm{Pro}(\mathcal{S}) \arrow[r, bend left, "\tau_{<\infty}"] & \mathrm{Pro}(\mathcal{S}_{<\infty}) \arrow[l, hook] \end{tikzcd}$$

has a left adjoint $\tau_{<\infty}$ given by the extension of the functor

$$\mathcal{S} \to \mathrm{Pro}(\mathcal{S}_{<\infty}), \quad X \mapsto \underset{\substack{\uparrow \\ \text{inverse system} \\ \text{given by Postnikov} \\ \text{tower}}}{\{\tau_{\leq n}(X)\}_{n \geq -2}}$$

Proposition $\Pi_{<\infty}^{ét}(\mathrm{Spec}(\mathbb{Z}))$ is trivial

proof n-truncated objects are hypercomplete.

We have thus shown

$$\tau_{\leq n}\big(\pi^{\acute{e}t}(\mathrm{Spec}\,\mathbb{Z})\big) =: \Pi^{\acute{e}t}_{\leq n}(\mathrm{Spec}\,\mathbb{Z}) \simeq *$$

The claim follows. □

It is unclear what more information the non-protruncated shape has to offer.

reference: "Exodromy" Barwick, Haine, Glasman