Γ-gp

rk: (1) If $1 \to G' \to G \to G'' \to 1$ is a s.e.s. of Γ-gps, then

$1 \to G'^{\Gamma} \to G^{\Gamma} \to G''^{\Gamma} \to$
$\to H^1(\Gamma, G') \to H^1(\Gamma, G) \to H^1(\Gamma, G'')$

· If $G' \triangleleft G$ is abelian $\boxed{\text{(and central)}}$

then

$\ldots \longrightarrow H^2(\Gamma, G')$ (?)

(2) If $S : \mathcal{C}^{op} \to \mathrm{Cat}$ (stack)
(fields)^{op}

k/k_0 - Γ-Gal-cover

& $X \in S(k)$.

$\rightsquigarrow$ $\boxed{1 \to \mathrm{Aut}(X) \to S\mathrm{Aut}(X) \to \Gamma}$

a k_0-form of $X \rightsquigarrow$ right splitting $\Gamma \to S\,\mathrm{Aut}(X)$

$\exists$? a form?: obstraction $H^2(\Gamma, \mathrm{Aut}(X))$ checking this.

Recall: fix a k_0-form X_0 of X.

$$H^1(\Gamma, \mathrm{Aut}(X)) = \frac{\{k_0\text{-forms of } X\}}{\text{iso}} \quad \text{"} \quad \frac{\{\text{semi-actions}\}}{\text{conj. by } \mathrm{Aut}\,X}$$

Ex: let $\mathcal{C} = \left(\mathrm{Alg}_{k_0}^{\text{fin, sep}}\right)^{op}$, k/k_0 fin. Gal.

$S_1 := \otimes$ Proj. Sch.

$S_2 := \mathrm{Alg}_{k_0}/\_$

$X_0^1 = \mathbb{P}^{n-1}_{k_0}$

$X_1 \in S_1(k)$ "$\mathbb{P}_k^{n-1}$"

$X_2 \in S_2(k)$ "$M_n(k)$"

$X_0^2 = M_n(k_0)$

Note: $\boxed{\mathrm{Aut}_{S_1(k)}(X_1) = \mathrm{Aut}_{S_2(k)}(X_2) = PGL_n(k)}$ $\ni A \mapsto (B \mapsto ABA^{-1})$

$A \mapsto \left([x_0 : \dots : x_{n-1}] \mapsto A[x_0, \dots, x_{n-1}]\right)$

Thm/Def: Let A be a k_0-v.s. w/ a binary op. TFAE (are called a CSA)

(1) $A \otimes_{k_0} k \cong M_n(k)$ for $k = \bar{k}_0$.

(1') $\exists F/k_0$ fin. s.t. $A \otimes_{k_0} F \cong M_n(F)$.

(2) $A \otimes_{k_0} k_0^{sep} \cong M_n(k_0^{sep})$.

(2.5) $\exists F/k_0$ fin. sep./Gal. ext. s.t.

$A \otimes_{k_0} F \cong M_n(F)$.

(3) A is an assoc. n^2-dim alg. s.t. $\mathrm{Cent}(A) = k_0$, A is simple.

Adam: (3) $\Leftrightarrow$ (1)

(1) $\Leftarrow$ (2)

Blackbox:

~~(1) $\Rightarrow$ (2) an~~

(?)

anything $\Rightarrow$ (2)

$\Gamma = Gal(k/k_0)$

Cor: $H^1(\Gamma, PGL_n) =$

$= \{ k_0\text{-forms of } \mathbb{P}^{n-1}_k \} =$

$= \{ k_0\text{-forms of } M_n(k) \} =$

$= \{ n^2\text{-dim CSA-s } A_0 / k_0 : A_0 \otimes_{k_0} k \cong M_n(k) \}$

Cor: $\boxed{H^1(k_0, PGL_n)} \cong \varinjlim_{F/k_0 \text{ fin}} H^1(Gal(F/k_0), PGL_n)$

$\boxed{= \{ n^2\text{-CSAs}/k_0 \} / \text{iso}.}$

Obs: If $n | m$: $PGL_n \to PGL_m$

$m = n \cdot k$ $\quad A \mapsto \underbrace{\begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}}_{k}$

$\rightsquigarrow H^1(k_0, PGL_n) \to H^1(k_0, PGL_m)$

$[A] \longmapsto [M_k(A)]$

Cor: $\boxed{Br(k_0) = \varinjlim_{(n \in \mathbb{N}, |)} H^1(k_0, PGL_n)}$

Note: $1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$

$\rightsquigarrow$ $\overbrace{0 \to 0}^{\text{Hilbert 90}} \to H^1(\mathrm{Gal}(F/k_0), PGL_n)$

$H^1(\mathrm{Gal}(F/k_0), \mathbb{G}_m) \quad H^1(\mathrm{Gal}(F/k_0), GL_n) = 0$

$\hookrightarrow H^2(\mathrm{Gal}(F/k_0), \mathbb{G}_m)$

taking $\varinjlim_{F/k_0 \text{ fin. Gal.}}$

$\rightsquigarrow$ $\boxed{H^1(k_0, PGL_n) \hookrightarrow H^2(k_0, \mathbb{G}_m)}$

taking $\varinjlim_{(n|m, 1)}$

$\rightsquigarrow$ $\boxed{Br(k_0) \hookrightarrow H^2(k_0, \mathbb{G}_m).}$

If $n|m$: $(m = n \cdot k)$

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & GL_n \arrow[r] \arrow[d] & PGL_n \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r] & GL_m \arrow[r] & PGL_m \arrow[r] & 1
\end{tikzcd}$$

$A \mapsto \begin{pmatrix} A & & \\ & \ddots & \\ & & A \end{pmatrix}$

$\Gamma = \mathrm{Gal}(F/k_0)$

surj: Note: F/k_0 - fin. Gal.

$F[\Gamma] := \{ \sum_{g\in\Gamma} a_g \cdot g \mid a_g \in F \}$

sum: as expected.

product: $(\sum a_g \cdot g)\cdot(\sum b_g \cdot g) =$

$= \sum_g \left(\sum_{h} a_h \cdot {}^h b_{h^{-1}g} \right) \cdot g.$

$$\boxed{\begin{aligned} &F \times_{k_0} F[\Gamma] = \\ &= F[\mathrm{Fun}(\Gamma,\Gamma)] \\ &\text{“}\mathrm{Func}(\Gamma, F)\text{”} \\ &\qquad\quad \| \\ &F \times_{k_0} F \cong \prod_{g\in\Gamma} F \end{aligned}}$$

observation: $F \times_{k_0} F[\Gamma] \cong M_n(F)$.

$H^2(k_0, \mathbb{G}_m) \to \mathrm{Br}(k_0)$: “WLOG $k_0^{sep} = F$ fin/k_0”

let $c : \Gamma \times \Gamma \to \mathbb{G}_m(k_0^{sep})$ be a 2-cocycle ~~$\Gamma = \mathrm{Gal}(k_0^{sep}/k_0)$~~

We def. an alg. by:

$A^c = \mathrm{span}_F \{ e_g \mid g \in \Gamma \}$ w/ mult.:

(w/ action of Γ on coeff.) $(n = |\Gamma|)$

$\boxed{n = [F : k_0]}$

$e_g \cdot e_h = c(g,h) \cdot e_{gh}$

ex.: for $c \equiv e$

$A^c = F[\Gamma]$

obs: $\boxed{A \times_{k_0} F \cong M_n(F)}$ is of F-alg. w/ a Γ-action

as v.s. w/ binary op.

$\rightsquigarrow [A^c] \in H^1(\mathrm{Gal}(F/k_0), PGL_n)$

$\downarrow$

~~$H^1(k_0$~~ $\mathrm{Br}(k_0)$

$[c] \quad H^2(k_0, \mathbb{G}_m)$.

$c \sim c'$