Goal: $Br(K) \cong H^2(K, \mathbb{G}_m)$

Goal 2:

$X \in S(F)$

$H^1(\mathrm{Gal}(F/K), \mathrm{Aut}(X))$

$\{K\text{-forms of } X\}$

Motivation:

(1) Alg. geo: X/k, k/k_0 - fin. Gal.

Q: (1) $\exists$? $X_0/k_0 : X_0 \times_{k_0} k \cong X$.

(2) If $\exists$, how many?

(2) Topo: $\tilde{P} \to \tilde{B}$ prin. bundle, $\tilde{B} \to B$ fin. normal.

(1) $\exists$? $P \to B$ s.t. $P \times_B \tilde{B} \cong \tilde{P}$

(2) If $\exists$, how many?

Not: Γ - fin. gp.

$\mathcal{C}$ - base cat

(usually a site)

think $\boxed{\mathcal{C} \simeq \left(\mathrm{Alg}^{\mathrm{fin,sep}}_{/k_0}\ \text{retale}\right)^{op}}$

$\mathcal{S} : \mathcal{C}^{op} \to \mathrm{Cat}$

pseudofunctor.

(usually a stack)

obj. of $\mathcal{C}$ will be denoted

$k, k_0, \ldots$

ex: $\mathcal{S} = \mathrm{QCoh}, \mathrm{Coh}, \mathrm{VB}, \mathrm{Sch}/-, \mathrm{CSA}(-), \ldots$

$\boxed{\mathrm{Sch} = \text{q. proj. sep. fin type.}}$

$\forall \gamma \in \Gamma$
$k \xrightarrow{\gamma} k$
$\searrow k_0 \swarrow$

__Def:__ A mor. $k \xrightarrow{u} k_0$ in $\mathcal{C}$ "with a Γ-action on k over k_0" is called a Γ-Gal. mor.

if $\boxed{\underline{\Gamma \times k} \xrightarrow{(\mathrm{id}, u)} k \times_{k_0} k}$

$\underline{\Gamma \times k} = \coprod_{g \in G} k$

$X_0 \in S(k_0)$

$X_0 \times_{k_0} k :=$
$= S(u)(X_0)$

furthermore if u is a cover we say it's a Γ-cover.

S pseudofunctor

$\leadsto \Gamma \curvearrowright S(k)$ a pseudo-action.

$\forall \gamma \in \Gamma \leadsto S(k) \to S(k)$
$X \mapsto {}^{\gamma}X$
$f \mapsto {}^{\gamma}f$.

${}^{\gamma}\gamma$-conj. object / mor?

$S = Sch/\_$.

$\forall \gamma \in \Gamma$ ${}^{\gamma}X := X$ as a scheme, w/ structure map $X \to \mathrm{Spec}\,k \xrightarrow{(\gamma^*)^{-1}} \mathrm{Spec}\,k$

$(= \text{p.b. along } \gamma^*)$

Def: A γ-semi-auto of $(\gamma \in \Gamma)$

$X \in S(k)$ is

an iso $\boxed{{}^{\gamma}X \xrightarrow{\sigma_\gamma} X}$.

~~Semi~~

if $\sigma_\alpha, \sigma_\beta$ are α/β-semi auto of X we def:

$$\sigma_\alpha \cdot \sigma_\beta := \sigma_\alpha \circ {}^{\alpha}(\sigma_\beta)$$

$$\begin{tikzcd} {}^{\alpha}({}^{\beta}X) \arrow[r] \arrow[d, "\wr"', "\text{nat. iso}"] & X \\ {}^{\alpha\beta}X \arrow[ur, "\sigma_\alpha \cdot \sigma_\beta"'] & \end{tikzcd}$$

pseudo funct. $\Rightarrow$ a group $\boxed{SAut(X)}$.

Ex: If $\mathcal{C} = \text{fields}^{op}$, $S = \mathrm{QCoh}$, a γ-semi auto is a k_0-linear auto satisfying

$T(\lambda v) = \gamma(\lambda) T(v)$ for $\lambda \in k$.

"More canonically": A γ-semi auto is an equiv. class of an iso $\gamma({}^{\beta}X) \to {}^{\beta}X$ up to action of Γ;

A semi-action of Γ on X is a homo $\Gamma \to \mathrm{SAut}(X)$ s.t.

$\gamma \mapsto \sigma_\gamma$

σ_γ is a γ-semi-auto $\forall \gamma \in \Gamma$.

$\boxed{\text{ex: If } X_0 \in S(k_0) \text{, then we get a semi-action on } X = X_0 \times_{k_0} k.}$

Fact/Cor: If $k \xrightarrow{u} k_0$ is Γ-Gal, ~~[illegible]~~ then:

$\boxed{\mathrm{Desc}(u, S) \xrightarrow{\sim} S(k)^{h\Gamma}:}$

ob: $\boxed{\{(X \in S(k), \underbrace{\sigma : \Gamma \to \mathrm{SAut}(X)}_{\text{semi-action}})\}}$

Mor: compatible maps.

Cor: If S satisfies descent w.r.t. u

$\boxed{S(k_0) \xrightarrow{\sim} S(k)^{h\Gamma} =} \{X \in S(k) + \text{semi-action}\}$

Def: Let $X \in S(k)$. A **k_0-form** of X is a $X_0 \in S(k_0)$ together with iso' $X_0 \times_{k_0} k \xrightarrow{\sim} X$ ($\in S(k)$).

$\{k_0\text{-forms of } X\}/\text{iso'}$

descent $\Rightarrow$ $\wr\|$

$\{\text{semi-actions on } X\}/\text{conj. by } \mathrm{Aut}(X) \subseteq \mathrm{SAut}(X)$

Assume X admits some k_0-form. Fix such a form.

Note: This induces a semi-action on X. $\Gamma \to \mathrm{SAut}(X)$

but also, this induces an action of Γ on $\mathrm{Aut}(X)$:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \wr\| & & \wr\| \\ X_0 \times_{k_0} k & \xrightarrow{\sim} & X_0 \times_{k_0} k \end{array}$$

$\mathrm{Aut}(X) \trianglelefteq \mathrm{SAut}(X)$

$\exists$ form

ex: $\mathcal{C} = (\text{fields}/k_0)^{op}$, $S = \mathrm{QCoh}$

$\gamma \in \Gamma$

$A \in GL_n(k)$:

$\rightsquigarrow {}^{\sigma}A \in GL_n(k)$

Thm:

$$H^1(\Gamma, \mathrm{Aut}(X)) \cong \{k_0\text{-forms of } X\}$$

Def: If Γ-gp, G — Γ-gp.

$$Z^1(\Gamma, G) := \{ c : \Gamma \to G,\ \gamma \mapsto c_\gamma : c_{\gamma\delta} = c_\gamma \cdot {}^{\gamma}c_\delta \}$$

$\underline{c \sim c'}$ $:\Leftrightarrow \exists b \in G : b \cdot c'_\gamma = c_\gamma \cdot {}^{\gamma}b \quad \forall \gamma \in G$.

$$Z^1(\Gamma, G)/\sim \; =: H^1(\Gamma, G)$$

Fact/Example: If $\mathcal{U} = \{\mathrm{Spec}\, F \xrightarrow{\text{Fin Gal.}} \mathrm{Spec}\, k\}$, $\mathcal{G}$ - (etale) sheaf of grps. put $\boxed{G = \mathcal{G}(\mathrm{Spec}\, F)}$ Then:

$$\check{H}^1_{et}(\mathcal{U}, \mathcal{G}) = H^1(\Gamma, G).$$

$SAut(X) \to \Gamma$
σ-semi-auto $\mapsto \gamma$

<u>Pf</u>: n.t.p. $\dfrac{\text{semi-actions}}{\text{conj. by } Aut(X)} = H^1(\Gamma, Aut(X))$

given $c \in Z^1(\Gamma, Aut(X))$ def.

a new semi-action by?

$\sigma'_\gamma = c_\gamma \circ \sigma_\gamma$

$\sigma_\gamma : \Gamma \to SAut(X)$ from the form X_0

given $\sigma' : \Gamma \to SAut(X)$

$c_\gamma = \sigma'^{\pm 1}_\gamma \circ \sigma^{\mp 1}_\gamma \rightsquigarrow \in Z^1(\Gamma, Aut(X))$

$\rightsquigarrow$ bijection. [redacted]

<u>ex</u>: $S^1 \xrightarrow{\times 2} S^1$, E-triv. rk 1 v.b. $e = top$, $f = \cdot 1/3$

$Aut(\underline{1}) = Map(S^1, GL_1(\mathbb{R})) \simeq S^1 \times \mathbb{Z}_2$

C_2 ⟲ by pre comp. w/ antipodal map.

$|H^1(C_2, Aut(\underline{1}))| = 2$

<u>cor</u>: since every line bundle on S^1 becomes triv. after this p.b.

$\Rightarrow |\{\text{line bundles on } S^1\}/\sim| = 2$.

② $\mathcal{C} = \text{fields}^{op}$ S = v.s. + non-deg. symm. form.

$\mathbb{C}/\mathbb{R}$ $\Gamma \cong C_2$ $\mathrm{Aut}(\mathbb{C}^n, S) = PO_n$

$|H^1(\mathbb{R}_{et}, PO_n)| = n+1$

by Sylvester's thm.

Def: $H^1(k_0, G(k)) =$

$$= \varinjlim_{\substack{k/F/k_0 \\ \text{fin}}} H^1(\mathrm{Gal}(F/k_0), G(k)^F)$$

<u>next time:</u>

$H^1(k_0, PGL_n) = \{k_0\text{-forms of } \mathbb{P}^{n-1}\}$

$= \{n^2\text{-dim CSA's}/k_0 \text{ upto iso}\}$.

$\subset Br(k_0)$.