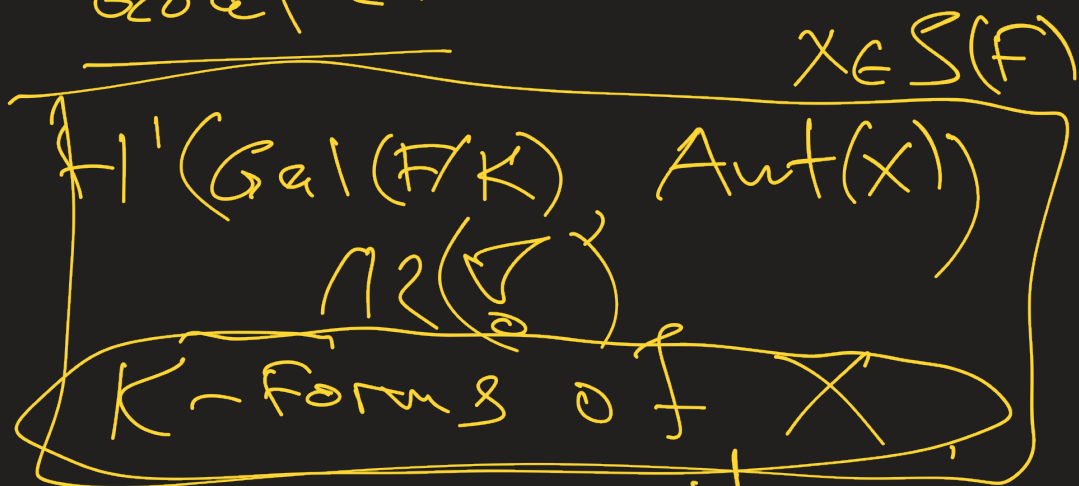


Goal 1: $Br(K) \cong H^2(K, \mathbb{G}_m)$

Goal 2:



Motivation: k/k_0 -fin. Gal.
(1) Alg. geo: X/k

Q: (1) $\exists X_0/k_0: X_0 \times_{k_0} k \cong X$.

(2) If \exists , how many?

(2) $\mathcal{P} \rightarrow B$ prin. bundle, $\bar{B} \rightarrow B$ fin. normal.

(1) $\exists \mathcal{P} \rightarrow B$ s.t. $\mathcal{P} \times_B \bar{B} \cong \mathcal{P}$

(2) If \exists , how many?

Note: Γ -fin. gp.
 \mathcal{C} -base cat

(usually a site)

think $\mathcal{C} = (\text{Alg}_{\Gamma, \text{sep}}^{\text{fin}})^{\text{op}}$
 \mathcal{G}_k retake

$\mathcal{S}: \mathcal{C}^{\text{op}} \rightarrow \text{Cat}$
pseudofunctor.

(usually a stack)

obj. of \mathcal{C} will be denoted

k, k_0, \dots

ex: $\mathcal{S} = \mathcal{Q}(\text{coh}, \text{coh}, \text{vB}, \text{Sch}/-, \text{CSA}(-1),$

$\text{Sch} = \text{g. proj. sep. Rim type.}$



Def: A var. $k \xrightarrow{\sigma} k_0$ in \mathcal{C} with a Γ -action $\hat{\sigma}$ on k over k_0 is called a Γ -Gal. var.

$$\text{if } \Gamma \times k \xrightarrow{(\text{id}, \sigma)} k \times_{k_0} k$$

$$\begin{array}{l} x_0 \in S(k_0) \\ x_0 \times_{k_0} k := \\ = S(\sigma)(x_0) \end{array}$$

$$k \sqcup k$$

g.c.f.
furthermore

if σ is a cover we say it's a Γ -cover.

S pseudofunctor

$\omega: \Gamma \rightarrow S(k)$ a pseudo-~~action~~ action.

$$\begin{array}{ccc} \forall \sigma \in \Gamma & \omega & S(k) \rightarrow S(k) \\ & & \begin{array}{ccc} X & \mapsto & \sigma X \\ k & \mapsto & \sigma k \end{array} \end{array}$$

γ -conj. object / mon[?]

$$S = S \circ h / -$$

$\forall \sigma \in \Gamma$ $\sigma_X := X$ as a scheme,
of structure map $X \rightarrow \text{Spec } k \xrightarrow{(\sigma^*)^{-1}} \text{Spec } k$

(= p.b. along σ^*)

Def: A σ -semi-auto of $(\sigma \in \Gamma)$

~~is~~ $X \in \mathcal{S}(k)$ is

an iso $\begin{matrix} \sigma \\ X \xrightarrow{\sim} X \end{matrix}$

~~A semi-auto~~

if σ, σ_β are α/β -semi auto of X we def:

$$\sigma_\alpha \cdot \sigma_\beta := \sigma_\alpha \circ \alpha(\sigma_\beta)$$

$$\begin{matrix} \alpha(\beta(X)) & \xrightarrow{\quad} & X \\ \text{nat. iso} \downarrow \alpha_\beta & \nearrow \sigma_\alpha \cdot \sigma_\beta & \\ \beta(X) & & \end{matrix}$$

pseudo-funct. \Rightarrow a group

$\boxed{\text{SAut}(X)}$

Ex:

If $\mathcal{L} = \text{fields}^{\text{op}}$

$$\mathcal{S} = \mathbb{Q}(dx)$$

a σ -semi auto is a k_0 -linear auto satisfying

$$\begin{aligned} T(\lambda v) &= \\ &= \sigma(\lambda) T(v) \\ &\text{for } \lambda \in k. \end{aligned}$$

"More canonically": A σ -semi auto

is an equiv. class of an iso

$$\sigma(\beta_X) \rightarrow \beta_X \text{ upto action of } \Gamma$$

A semi-action of Γ on X is a homomorphism $\Gamma \rightarrow \text{SAut}(X)$ s.t.

$\gamma \mapsto \sigma_\gamma$
 σ_γ is a γ -semi-auto
 $\forall \gamma \in \Gamma$

ex: If $X_0 \in S(k_0)$, then we get a semi-action on $X = X_0 \times_{k_0} k$.

Fact/Cor: If $k \xrightarrow{u} k_0$ is Γ -Gal, ~~and~~
~~S satisfies descent w.r.t. u~~ then:

$\text{Desc}(u, S) \cong S(k)^{\text{h}\Gamma}$

Def: $\{ (X \in S(k), \sigma: \Gamma \rightarrow \text{SAut}(X)) \}$ semi-action

Def: compatible maps.

Cor: If S satisfies descent w.r.t. u

$S(k_0) \cong S(k)^{\text{h}\Gamma} = \{ X \in S(k) + \text{semi-action} \}$

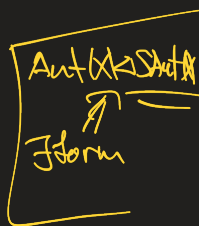
Def: Let $X \in S(k)$. A k_0 -form of X is a $X_0 \in S(k_0)$ together with an iso $X_0 \otimes_{k_0} k \cong X|_{(-S(k))}$.

$\{k_0\text{-forms of } X\} / \cong$
 $\xrightarrow{\text{descent}} \{ \text{semi-actions on } X \} / \text{conj. by } \text{Aut}(X) \subseteq \text{Aut}(k)$

Assume X admits some form. Fix such a form.

Note! This induces a semi-action on X , Γ - $\text{Aut}(X)$

but also, this induces an action of Γ on $\text{Aut}(X)$:



ex: $C = (\text{field}/k)$
 $S = \text{Dch}$
 $\Gamma \in \Gamma$
 $A \in \text{GL}_n(k)$
 $\mapsto \sigma A \in \text{GL}_n(k)$

Then:

$$H^1(\Gamma, \text{Aut}(X)) \cong \{k_0\text{-forms of } X\}$$

Def: If Γ -gp, G - Γ -gp.

$$Z^1(\Gamma, G) := \{c: \Gamma \rightarrow G : c_{\gamma\sigma} = c_\sigma \cdot \sigma c_\gamma\}$$

$$c \sim c' : \Leftrightarrow \exists b \in G : b \cdot c'_\gamma = c_\gamma \cdot \sigma_b \quad \forall \gamma \in \Gamma$$

$$Z^1(\Gamma, G) / \sim =: H^1(\Gamma, G) \quad \text{Fin Gal.}$$

Fact/Example: If $\mathcal{U} = \{ \text{Spec } F \rightarrow \text{Spec } k \}$, \mathcal{G}_F -sheaf (etale) of grps. put $\mathcal{G} = \mathcal{G}_F(\text{Spec } F)$. Then:

$$\check{H}_{\text{et}}^1(\mathcal{U}, \mathcal{G}) = H^1(\Gamma, \mathcal{G})$$

$$\begin{array}{l} S\text{Aut}(X) \rightarrow \Gamma \\ \sigma\text{-semi} \\ \text{-auto} \mapsto \sigma \end{array}$$

PF n.t.p. semi-actions = $H^1(\Gamma, \text{Aut}(X))$
 conj. by $\text{Aut}(X)$

given $C \in Z^1(\Gamma, \text{Aut}(X))$ def.
 a new semi-action?

$$\sigma'_\gamma = C_\gamma \circ \sigma_\gamma$$

given $\sigma': \Gamma \rightarrow \text{Stuff}(X)$

$\sigma_\gamma: \Gamma \rightarrow \text{Aut}(X)$
 from the
 base X_0

$$C_\gamma = \sigma_\gamma^{-1} \circ \sigma'_\gamma \mapsto \in Z^1(\Gamma, \text{Aut}(X))$$

\leadsto bijection. ~~Stack~~

ex: $S \xrightarrow{\times 2} S'$, E -triv! v.b. $S = \mathbb{R}^3$ $e = \text{top}$

$$\text{Aut}(\underline{\mathbb{1}}) = \text{Map}(S', \text{GL}(R)) \cong S' \times \mathbb{Z}_2$$

$C_2 \mathbb{Q}$ by pre comp. w/ antipodal map.

$$|H^1(C_2, \text{Aut}(\underline{\mathbb{1}}))| = 2$$

cor: since every line bundle on S^1
 becomes triv. after this p.b.

$$\Rightarrow |\{ \text{line bundles on } S^1 \} / \sim} = 2.$$

② $\mathcal{C} = \text{fields}^{\text{op}}$ $\mathcal{S} = \text{v.s. + non-deg. symm. form.}$

$$\mathbb{C}/\mathbb{R} \quad \sqrt{\quad} \cong \mathbb{C} \quad \text{Aut}(\mathbb{C}, \mathcal{S}) = \text{PO}_n$$

$$\uparrow \text{H}'(\mathbb{R}\text{-Ret}, \text{PO}_n) = n-1$$

by Sylvestre's thm.

~~Def:~~ $\text{H}'(k_0, \mathcal{S}(A)) =$

$$= \varinjlim_{\substack{k/F/k_0 \\ f \in \mathcal{S}(A)}} \text{H}'(\text{Gal}(F/k_0), \text{Br}(k)^F)$$

next time:

$$\text{H}'(\mathbb{R}/k_0, \text{PGal}) = \{k_0\text{-forms of } P^{n-1}\}$$

$$= \{n\text{-dim CSTS}/k_0 \text{ up to iso}\}.$$

$$+ \text{Br}(k_0).$$