Exercises

Exercise 1: Let $X \xrightarrow{f} Y$ be a separated morphism of schemes with X irreducible and Y reduced which is locally an isomorphism, i.e.

$$\forall\, x \in X \ \exists\, U \hookrightarrow X \quad \text{s.t.} \quad f|_U : U \xrightarrow[f]{\cong} f(U) \hookrightarrow Y$$

Show f is an open embedding.

Exercise 2: Suppose we have a commutative diagram of schemes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d, "g"] \\ & S \end{tikzcd}$$

formally unramified

formally étale

with the given blue conditions.

Show that f is formally étale.

Exercise 3: Let X be a sober space (every irreducible closed subset has a generic point). Show that the topology of X can be recovered from the associated site $\mathrm{Open}(X)$

view this purely as a poset, i.e. you do not have the point set information of each object

Exercise 4: Show that there are spaces that can not be recovered from a Grothendieck topology.

Hint: By the previous Exercise they will not be sober.

Exercise 5: (a) Let Z be a G-set. Show that the representable functor $h_Z = \mathrm{Hom}_G(-, Z)$ is a sheaf in T_G.

(b) Finish the proof of the equivalence

$$\mathrm{Sh}(T_G) \simeq G\text{-Sets}$$

Exercise 6: Let X be a scheme and $\mathcal{F}$ a sheaf on X_{zar}. Assume $\mathcal{F}$ satisfies the sheaf axiom for single morphism fpqc coverings $V \longrightarrow U$. Show that $\mathcal{F}$ is a sheaf (of sets) on X_{fpqc}.

Exercise 7: Assume the representable functor h_Z is a sheaf on X_{fpqc} for all affine $Z \in Sch/_X$. Show that h_Z is a sheaf for any $Z \in Sch/_X$.

Exercise 8: Show that $\mathcal{O}_X^{\acute{e}t}$, given by $\mathcal{O}_X^{\acute{e}t}(U) = \mathcal{O}(U)$

for $U \to X$ étale, is an étale sheaf.

Exercise 9: Let C be a curve over a field k. Show that

$$(C \setminus p) \amalg \operatorname{Spec} \mathcal{O}_{C,p} \longrightarrow C$$

is an fpqc cover for any point p of C.

Exercise 10: Let $\mathcal{C}$ be a site and $\mathrm{Cov}(X)$ the coverings of $X \in \mathcal{C}$. Show that any map of coverings $\mathcal{U} \to \mathcal{V}$ induces a map on Čech cohomology

$$\check{H}^*(\mathcal{U}; \mathcal{F}) \longrightarrow \check{H}^*(\mathcal{V}, \mathcal{F}) \qquad (\text{for any presheaf } \mathcal{F}).$$

Moreover, any map of coverings induces the same map.

Exercise 11: Show that

$$\check{H}^q(\mathcal{U}; \mathcal{F}) \cong R^q \check{H}^0(\mathcal{U}, \mathcal{F})$$

as functors $\mathcal{P}Ab(\mathcal{C}) \to Ab$.

Exercise 12: Let $A \to B$ be flat, $M \in Mod_A$ finitely presented, $N \in Mod_A$. Show

$$Hom_A(M, N) \otimes_A B \xrightarrow{\cong} Hom_B(M \otimes_A B, N \otimes_A B)$$

Exercise 13: Let $M \in Mod_A$. Show TFAE:

(1) M is finitely presented and projective

(2) M is finitely generated projective

(3) M is a direct summand of A^n for some n

(4) M is finitely presented and flat

[5] M is Zariski-locally free