Assumptions

$k = \bar{k}$; $(n, \operatorname{char} k) = 1$

Recap: $H^*(\mathbb{G}_m)$

X nonsing, connected $\Rightarrow H^*(X_{ét}, \mathbb{G}_m) \cong \begin{cases} k^\times & *=0 \\ \operatorname{Pic}(X) & *=1 \\ 0 & *>1 \end{cases}$

Idea of pf.

LES of Weil ex. seq. of sheaves on $X_{ét}$:

$$0 \to \mathbb{G}_{m,X} \to g_* \mathbb{G}_{m,K} \to \operatorname{Div} \to 0$$

(under $g_*\mathbb{G}_{m,K}$): Leray s.s.
- ét. shvs on pt.
- Tsen's
- $Br = H^2$
- Galois cohom.

For $H^q(\operatorname{Spec} K, \mathbb{G}_m) = 0$ if $q>0$ and $\deg tr(K/k) = 1$

(under Div): $\cong \bigoplus_z i_{z*}\mathbb{Z}$ and Leray ss

Today: $H^*(\mu_n)$ (Recall: μ_n is sheaf rep. by $X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}[t]/t^n-1$ or $U \mapsto \{x \in \Gamma(U, \mathcal{O}_U) \mid x^n = 1\}$)

- X complete, conn., nonsing.
- relative to closed $x \to X$; X nonsing
- w/cpct support, X conn., regular

§1: Thm: X complete, conn., nonsing, genus g

$$\Rightarrow H^*(X_{ét}, \mu_n) \cong \begin{cases} \mu_n(k) & *=0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & *=1 \\ \mathbb{Z}/n\mathbb{Z} & *=2 \\ 0 & *>2 \end{cases}$$

Pf: Step 1: Kummer ex. seq.:

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow[t \mapsto t^n]{n} \mathbb{G}_m \to 0$$

Idea: check stalks at geom pts $\bar{x} \to X$; ie for strict hensel A want

$$0 \to \mu_n(A) \to A^\times \xrightarrow{t \mapsto t^n} A^\times \to 0 \quad \text{exact}$$

(✓ ; ✓ by def)

at 2nd $A^\times$: want $T^n = a$ to have soln for all $a \in A$

on res. field, $\frac{d}{dT}(T^n - a) = nT^{n-1} \neq 0$ (as $(n,p)=1$)

so have simple root that can lift via Hensel to A.

Step 2: ker and coker of $\operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X)$

$\operatorname{Div}^0(X)$ = divisors of deg 0

As X complete, $K := k(X)^\times \xrightarrow{div} \operatorname{Div}^0(X)$ (ratl. fcts)

$\operatorname{Pic}^0(X) := \operatorname{Div}^0(X)/\operatorname{im}(div) \to \operatorname{Div}(X)/\operatorname{im}(div) = \operatorname{Pic}(X)$

Lemma: (1) $0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \to \mathbb{Z} \to 0$

(2) $\ker(\operatorname{Pic}^0(X) \xrightarrow{n} \operatorname{Pic}^0(X)) = (\mathbb{Z}/n\mathbb{Z})^{2g}$

coker (— " —) =

Pf. (1) Consider $k(X)^\times \xrightarrow[f]{div} \operatorname{Div}^0(X) \xhookrightarrow[g]{} \operatorname{Div}(X)$

Then ker-coker seq:

$$\ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(g \circ f) \to \operatorname{coker}(g) \to 0$$
$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \to \mathbb{Z} \to 0$$

(= degree)

(2) ommitted.

$\underline{k = \mathbb{C}}$: $\operatorname{Pic}^0(X) \simeq \mathbb{C}^g/\Lambda$ ← full rank lattice from holom diffs & gens of $H_1(X(\mathbb{C}), \mathbb{Z})$

$\downarrow n \qquad \downarrow$ (ker $= (\mathbb{Z}/n\mathbb{Z})^{2g}$)

$\operatorname{Pic}^0(X) \simeq \mathbb{C}^g/\Lambda$

$\downarrow$

0

$\underline{k \neq \mathbb{C}}$: $\operatorname{Pic}^0(X) \simeq k$-pts of $\operatorname{Jac}(X)$ (ab. var. of dim $= g$) □

Now for $\operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X)$

$$0 \to (\mathbb{Z}/n\mathbb{Z})^{2g} \to ?? \to 0$$
$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \to \mathbb{Z} \to 0$$
$$\downarrow n \qquad \downarrow n \qquad \downarrow n$$
$$0 \to \operatorname{Pic}^0(X) \to \operatorname{Pic}(X) \to \mathbb{Z} \to 0$$
$$0 \to ?? \to \mathbb{Z}/n\mathbb{Z} \to 0$$

(Lemma)

Hence, $\ker(\operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X)) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$

and coker(— " —) $= \mathbb{Z}/n\mathbb{Z}$

Step 3: LES of Kummer seq

$$0 \to H^0(X, \mu_n) \to k^\times \xrightarrow{(\cdot)^n} k^\times \to H^1(X, \mu_n) \to \operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X) \to H^2(X, \mu_n) \to 0 \to 0$$

($k^\times = H^0(X, \mathbb{G}_m)$; surj. as $k = \bar{k}$; $\operatorname{Pic}(X) = H^1(X, \mathbb{G}_m)$)

$H^0(X, \mu_n) \cong \ker(k^\times \xrightarrow{(\cdot)^n} k^\times) = \mu_n(k)$

$H^1(X, \mu_n) \cong \ker(\operatorname{Pic}(X) \xrightarrow{n} \operatorname{Pic}(X)) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ — Natural

$H^2(X, \mu_n) \cong \operatorname{coker}(— " —) \cong \mathbb{Z}/n\mathbb{Z}$ □

Rem: (Artin-Grot-Verd.) $k = \bar{k}$, X/k curve, $\mathcal{F} \in Sh(X_{ét})$ torsion

(1) $H^q(X, \mathcal{F}) = 0$ for $q > 2$

(2) — " — $q > 1$ if X affine

Idea: a) Trace methods
b). Ch 15 in Milne on cohom dimension ← more general

§2 H^* relative to a closed point

Prop: X nonsing, $x \to X$ closed

$$H^r_x(U, \mu_n) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & r=2 \\ 0 & \text{o.w.} \end{cases}$$

$Z \hookrightarrow X$ closed; $U = X \setminus Z$. $H^*_Z(X, \mathcal{F})$ defined as right der. fctrs. of $\Gamma_Z(X, \mathcal{F}) = \ker(\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F}))$ ← sections w/support on Z.

Pf: $R = \mathcal{O}^h_{X,x}$ ← henselization; $\operatorname{Spec} R \xrightarrow{ét} X$

1) Excise to $(V = \operatorname{Spec} R, x)$: excision (ex?) $\Rightarrow H^r_x(U, \mu_n) \simeq H^r_x(\operatorname{Spec} R, \mu_n)$

Write $V = \operatorname{Spec} R$

2) compute $H^r_x(V, \mathbb{G}_m)$:

ex. seq. of $(V, V \setminus x)$ $\Rightarrow$

$$\dots \to H^r_x(V, \mathbb{G}_m) \to H^r(V, \mathbb{G}_m) \to H^r(V \setminus x, \mathbb{G}_m) \to \dots$$

($H^r(V, \mathbb{G}_m) = 0$ for $r \geq 1$ as V is spec of strictly henselian local ring [Stacks 03QO])

$$\Rightarrow H^{r-1}(V \setminus x, \mathbb{G}_m) \xrightarrow{\simeq} H^r_x(V, \mathbb{G}_m) \quad (r \neq 1)$$

($V \setminus x = \operatorname{Spec} K$ where $K = \operatorname{Frac}(R)$)

& [Lang] K is C_1 for R henselian DVR w/same conditions, so $H^{r-1}(K, \mathbb{G}_m) = 0$ for $r \geq 2$

$$\Rightarrow H^r_x(V, \mathbb{G}_m) \simeq \begin{cases} 0 & r \neq 1 \\ H^0(\operatorname{Spec} K, \mathbb{G}_m)/H^0(V, \mathbb{G}_m) \simeq K^\times/R^\times \simeq \mathbb{Z} & \end{cases}$$

($H^0(\operatorname{Spec}K, \mathbb{G}_m) = K^\times$, $H^0(V, \mathbb{G}_m) = R^\times$; R is DVR)

3) Kummer again

Now LES of Kummer $0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ for $H^*_x(V, -)$:

$$0 \to H^0_x(V, \mu_n) \to 0 \xrightarrow{n} 0 \to H^1_x(V, \mu_n) \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to H^2_x(V, \mu_n) \to 0 \to 0 \dots$$

$$\Rightarrow H^r_x(V, \mu_n) = \begin{cases} \mathbb{Z}/n\mathbb{Z} & r=2 \\ 0 & \text{o.w.} \end{cases}$$

§3 H^* w/cpct support

K/k function field $:= [K : k(T)] < \infty$

X/k regular curve if all stalks are DVR

(nonsing $\Rightarrow$ regular; $\Leftarrow$ if k perfect)

X/k reg. curve $\Rightarrow K = k(U)$ is fct. field $/k$

and $X \longrightarrow \{\text{DVRs } k \subset R \subset K \mid \operatorname{Frac}(R) = K\}$

$x \longmapsto \mathcal{O}_{X,x}$

is injection (uses "varieties are separated"; else double line)

Fct. field$/k$ $\longrightarrow$ conn. complete reg. curve

$K \mapsto$
- $X = \{\text{DVRs } k \subset R \subset K \mid — " — \}$ (w/ top. where prop. closed subsets are the finite ones)
- $\mathcal{O}_X = (U \mapsto \bigcap_{R \in U} R)$

U conn. reg. curve $/k$ $\rightsquigarrow X$ conn. reg. complete curve assoc. to $k(U)$

$\rightsquigarrow j : U \hookrightarrow X$

$u \longmapsto \mathcal{O}_{U,u}$

Define $\boxed{H^r_c(U, \mathcal{F}) := H^r(X, j_! \mathcal{F})}$

($j_!$ = ext. by zero, is left adj. to $j^* : Sh(X_{ét}) \to Sh(U_{ét})$; if $U \hookrightarrow X$, then $(j_! \mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & x \in U \\ 0 & x \notin U \end{cases}$)

(1) $j_!$ exact $\Rightarrow$ (s.e.s. of sheaves $\rightsquigarrow$ LES in H^*)

(2) $j_!$ doesn't preserve injectives $\Rightarrow H^r_c(U, -) \neq \mathbb{R}^r H^0_c(U, -)$

Prop: U conn. regular

$$\Rightarrow H^2_c(U, \mu_n) \xrightarrow{\simeq} \mathbb{Z}/n\mathbb{Z} \quad \text{canonical}$$

Pf: $U \xhookrightarrow{j} X$ (regular, complete), canonical inclusion

$\uparrow i$

$Z \quad (U = X \setminus Z)$

π_* = direct image

π^* = inverse image

With $\mu_n \in Sh(X_{ét})$:

$$0 \to j_! j^* \mu_n \to \mu_n \to i_* i^* \mu_n \to 0 \quad \text{exact}$$

LES $\Rightarrow$

$$\dots \to H^r_c(U, \mu_n) \to H^r(X, \mu_n) \to H^r(X, i_* i^* \mu_n) \to \dots$$

$H^r(X, i_* i^* \mu_n) \simeq H^r(Z, i^* \mu_n) = 0$ for $r > 0$ ← by Leray s.s. and $R^{>0} i_* = 0$ trick

$$\Rightarrow H^2_c(U, \mu_n) \xrightarrow{\simeq} H^2(X, \mu_n) \simeq \mathbb{Z}/n\mathbb{Z}$$ □