__Goal:__ $H^1(X_{et}, \mathbb{G}_m) = H^1(X_{zar}, \mathbb{G}_m) = \mathrm{Pic}(X)$.

__Descent:__ Let $\mathcal{C}, \mathcal{D}$ be (∞) cats, $F : \mathcal{C} \to \mathcal{D}$ an (∞) functor. Let ~~...~~

$$Y \xrightarrow{u} X \in \mathcal{C}$$

__Def:__ $\mathrm{Desc}(u, F) :=$

$$\operatorname{holim}\left(F(Y) \rightrightarrows F(Y \times_X Y) \mathrel{\substack{\to\\\to\\\to}} F(Y \times_X Y \times_X Y) \mathrel{\substack{\to\\\to\\\to\\\to}} \cdots\right)$$

__Def:__ ~~If $\mathcal{C}$ carries a top.~~ (1) we say F satisfies descent w.r.t. u if the natural map

$$F(X) \to \mathrm{Desc}(u, F) \text{ is a w.e.}$$

(2) If $\mathcal{C}$ carries a top. we say F is an (∞) sheaf/stack if it satisfies descent w.r.t. any covering family.

__Fact of life:__ If $\mathcal{D}$ is an n-cat, then we can truncate the above diagram at step $n(+1)$.

__Related Phenomena)__

"Let $\mathcal{C}, \mathcal{D}$ be n-cat/m-cat.

~~(1) If $m > n$, then ∞-functors $\mathcal{C} \to \mathcal{D}$ are "interesting"~~

(2) If $m < n$, the ∞-functors $\mathcal{C} \to \mathcal{D}$ factor through $\mathcal{C}$'s m-th homotopy cat."

<u>moral reason:</u>

"there are no maps from an n-connected space X to an $n+C$-truncated space Y."

examples:

(1) 1-desent: $\mathcal{D}$- cat.

$F : \mathcal{C} \to \mathcal{D}$ satisfies descent w.r.t. $u : Y \to X$ iff

$$F(X) \to F(Y) \rightrightarrows F(Y \times_X Y)$$

is an equalizer.

(2) 2-desent: If $\mathcal{D}$ is Cat.

then: $Y \xrightarrow{u} X$

Desc(u, F):

Ob: $F(Y) \underset{g}{\overset{f}{\rightrightarrows}} F(Y \times_X Y) \rightrightarrows F(Y \times_X Y \times_X Y)$

mor: compatible maps.

Comm. algebra:

- Recall: if $A \to B$ is f.f., then

$$0 \to A \to B \to B \otimes_A B \to \cdots$$

(In part., $\forall M \in \underline{A\text{-Mod}}$,

$$0 \to M \to M \otimes_A B \to B \otimes_A M \otimes_A B \to \cdots)$$

Prop: If $A \to B$ is flat, $M \in \underline{A\text{-Mod}}$ is fin. pres., and $N \in \underline{A\text{-Mod}}$, then:

$$\mathrm{Hom}_A(M,N) \otimes_A B \xrightarrow{\sim} \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B).$$

Prop: Let $A \to B$ be f.f., $M \in \underline{A\text{-Mod}}$, $N = B \otimes_A M$.

(1) M is f.g./A $\Leftrightarrow$ N is f.g./B
(2) M is f.p./A $\Leftrightarrow$ N is f.p./B
(3) M is a v.b. $\Leftrightarrow$ N is a v.b.

Recall: For $M \in \underline{A\text{-Mod}}$, TFAE:

$\boxed{\text{(1) } M \text{ is f.p. proj.}}$ (2) M is f.g. proj.
(3) M is direct summand of A^n for some $n \in \mathbb{N}$,
(4) M is f.p. flat
(5) M is Zariski-locally free.

Pf: (1) $\Rightarrow$ clear.

$\Leftarrow$ say $B\otimes_A M$ is gen. by $\{\sum_{j=1}^{n_i} b_j^i \otimes m_j^i\}_{i=1}^{m}$, then we claim $\{m_j^i\}$ generate M.

$$A^N \to M \to C \to 0 \quad /\otimes B$$

$$B^N \twoheadrightarrow B\otimes_A M \to \cancel{B\otimes_A C} \to 0$$

clearly surj. f.f. $\Longrightarrow$ $C = 0$.

(2) $\Rightarrow$: clear.

$\Leftarrow$: know: $B\otimes_A M$ is f.p.

by (1), M is f.g.

take a surj: $0 \to K \to A^{n} \to M \to 0 \quad /\otimes B$ (n finite)

$$0 \to (B\otimes_A K \Rightarrow) \to B^n \to B\otimes_A M \to 0$$

f.g. b.c. $B\otimes_A M$ is f.p.

by (1) again, K is f.g.

(3) $\Rightarrow$ clear : $M \oplus N \cong A^n$

$\rotatebox{90}{$\Rightarrow$}$

$(B \otimes_A M) \oplus (B \otimes_A N) \cong B^n$

$\Leftarrow$: know: $B \otimes_A M$ is f.p. proj.

$\Rightarrow$ f.p. $\xRightarrow{(2)}$ M is f.p.

Let $C^\bullet$ be an acyclic cx of A-modules.

By previous prop:

$\mathrm{Hom}_A(M, C^\bullet) \otimes_A B = \mathrm{Hom}_B(B \otimes_A M, B \otimes_A C^\bullet)$

is acyclic. proj. acyclic

By faithful flatness, $\mathrm{Hom}_A(M, C^\bullet)$ is acyclic.

Examples of descent:

(1) Let M- A-mod.

So M is a sheaf on $(A/\mathrm{Ring})^{op}_{\tau}$ with Zar ~~[illegible]~~

Pf: We need to show that $\forall \{f_i\} \subset A$ s.t. $(f_i) = A$,

$$0 \to M \to \prod_i M_{f_i} \rightrightarrows \prod_{i,j} M_{f_i f_j}$$

this is an immediate cor. of the $0 \to A \to B \to B \otimes_A B \to \_ \_$.

taking $\boxed{B = \prod_i A_{f_i}}$ (which is f.f.).

2-desent:

Prop: QCoh, Coh, Vect : $(Sch/S)^{op} \to Cat$ (Grpd)

is ~~en~~ fpqc stack.

($\Leftrightarrow$ it's a τ-stack $\forall$ $Zar \leq \tau \leq fpqc$)

"pf":

(1) Viktor: it's enough to

(?) check for affine stuff.

(2) $A \xrightarrow{f} B$ f.f. we have:

$u = Spec(f)$

$$\boxed{ModA \xrightarrow{F} Desc(u, QCoh)}$$

$$M \mapsto (B\otimes_A M \rightrightarrows B\otimes_A B\otimes_A M \rightrightarrows B^{\otimes 3}_{\otimes_A} M)$$

Define: $G : Desc(u, QCoh) \to ModA$ by:

$$\left(N \underset{f_2}{\overset{f_1}{\rightrightarrows}} B\otimes_A N \rightrightarrows B^{\otimes_A 2}\otimes_A N\right) \mapsto eq(f_1, f_2)$$

need to show: $\boxed{Id_{ModA} \cong G\circ F}$ ← from the exact seq.

$F\circ G \cong Id_{Desc(u,QCoh)}$ η

Idea: "construct a natural map. η

(2) η being an iso is fflat-local

(3) p.o. along $A\to B$ to have a retraction and use it.

descent for
• Coh, Vect_n follows immediately from descent for QCoh & the prop. from before.

<u>Cor (*)</u>: $\left\{ \begin{matrix} \tau\text{-vector bundles} \\ \text{of rk } n \end{matrix} \right\} / \text{iso}$

does not depend on τ

for $\mathrm{Zar} \subseteq \tau \subseteq \mathrm{fppf}$.

<u>Facts</u>:

(1) $\check{H}^1(\mathfrak{U}, G) \underset{\text{easy}}{\cong} \left\{ \begin{matrix} G\text{-principal bundles} \\ \text{trivialized by } \mathfrak{U} \end{matrix} \right\} / \sim$

(2) If G is abelian

$\check{H}^1(X, G) \cong H^1(X, G)$ maybe requires something.

(3) $\{ GL_n\text{-principal bundles} \} / \sim \ \underset{\text{easy}}{\cong} \ \mathrm{Vect}_n$