Cohomology of Curves - 1

The goal of this talk is to lay some foundations needed to compute $H^*(X_{et}, \mathbb{G}_m)$ when X is a non-singular projective curve over an algebraically closed field. Unfortunately, the computation is quite involved, so we won't get through all of it today. Instead, we'll set our sights on proving the following theorem

<u>Tsen's Theorem:</u> Suppose X is a curve over an algebraically closed field k with function field K. Then K is C_1 - i.e. quasi-algebraically closed.

Of course, a priori this seems at best tangentially related to the stated goal of computing $H^*(X_{et}, \mathbb{G}_m)$, so we'll begin by explaining why Tsen's Theorem is a key ingredient ~~in our~~ towards realizing this goal.

Recall, the function field of an irreducible curve is the residue field at the generic point, so let

$$g : \operatorname{Spec} K \longrightarrow X$$

denote the inclusion of the generic point. We then get a natural injection of étale sheaves

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow g_* \mathbb{G}_{m,K}$$

as follows: for $U \hookrightarrow X$ étale, $\mathbb{G}_{m,X}(U) = \mathcal{O}_U(U)^\times$

and $g_*\mathbb{G}_{m,K}(U) = \mathbb{G}_{m,K}(U \times_X \operatorname{Spec}(K)) = \mathcal{O}_{U \times_X \operatorname{Spec}(K)}(U \times_X \operatorname{Spec}(K))^\times$

<u>Exercise:</u> Identify $g_*\mathbb{G}_{m,K}(U)$ with meromorphic functions on U (reduce to the case that $U \hookrightarrow X$ is integral and affine, then chase definitions)

In the case that X is integral and regular, we'll identify the cokernel with the sheaf of Weil divisors ~ which we'll now define.

Weil Divisors: For any co-dimension 1 point $v \in X$ (so in particular $\mathcal{O}_{X,v}$ is 1 dimensional) and let $i_v : \mathrm{Spec}(k(v)) \to X$ be the natural map. Let X_1 denote the set of co-dimension 1 points. The sheaf of Weil divisors is then defined to be

$$D_X := \bigoplus_{v \in X_1} (i_v)_* \mathbb{Z}$$

Now if X is integral and regular, the maximal ideals $m_v \subseteq \mathcal{O}_{X,v}$ are principal, so we can choose generators t_v

$$m_v = \langle t_v \rangle$$

We can use these generators to define discrete valuations on $\mathrm{Frac}(\mathcal{O}_{X,v})$. First, for $a \in \mathcal{O}_{X,v}$ we can write

$$a = u t_v^n$$

for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}_{X,v}^\times$, and we send

$$a \longrightarrow \mathrm{ord}_v(a) = n$$

For a fraction $\frac{a}{b}, b \neq 0$ we define $\mathrm{ord}_v(\frac{a}{b}) = \mathrm{ord}_v(a) - \mathrm{ord}_v(b)$.
One easily checks this defines a valuation

$$K^\times \longrightarrow \mathbb{Z}$$

So we use the same construction to define a map of sheaves

$$g_* \mathbb{G}_{m,K} \longrightarrow D_X$$

Lemma: For X an integral regular curve, the sequence of étale sheaves

$$0 \to \mathbb{G}_{m,X} \to g_* \mathbb{G}_{m,K} \xrightarrow{\mathrm{div}} D_X \to 0$$

is exact.

Proof: The only non-obvious part is surjectivity $g_* \mathbb{G}_{m,K} \to D_X$. So fix a geometric point $\bar{x}$ and look at stalks

$$0 \to A^\times \to \mathrm{Frac}(A)^\times \xrightarrow{\mathrm{ord}} \bigoplus_{ht(p)=1} \mathbb{Z} \to 0$$

Since each prime of height 1 is actually principal, surjectivity is obvious

$\square$

Ok, so the short exact sequence reduces our study of $H^*(X_{\acute{e}t}, \mathbb{G}_m)$ to that of $H^*(X_{\acute{e}t}, g_*\mathbb{G}_{m,K})$ and $H^*(X_{\acute{e}t}, D_X) = \bigoplus_{v \in X_1} H^*(X_{\acute{e}t}, (i_v)_* \mathrm{Spec}(k(v)))$. Eventually, we will see that $H^*(X_{\acute{e}t}, g_*\mathbb{G}_{m,K}) = 0$ for $* > 0$ and $H^*(X_{\acute{e}t}, D_X) = 0$ for $* > 0$, which will prove

$$H^*(X_{\acute{e}t}, \mathbb{G}_m) = \begin{cases} \mathcal{O}_X(X)^* & * = 0 \\ \mathrm{Pic}(X) & * = 1 \\ 0 & \text{else.} \end{cases}$$

<u>Lemma:</u> $H^*(X_{\acute{e}t}, D_X) = 0$ for $* > 0$, if X is over an alg. closed field.

<u>Proof:</u> Since cohomology commutes with direct sums, we need only show that $H^*(X_{\acute{e}t}, (i_v)_* \mathbb{Z}) = 0$ for $* > 0$.

The Leray spectral sequence takes the form

$$H^p(X_{\acute{e}t}, R^q i_{v*} \mathbb{Z}) \Longrightarrow H^{p+q}(\mathrm{Spec}(k(v))_{\acute{e}t}, \mathbb{Z})$$

Since X is a curve, any co-dimension 1 point is closed ($\dim \mathcal{O}_{X,x} = \dim X$) and thus $k(v)$ is isomorphic to the base field - i.e. it's algebraically closed. Thus all sheaves on $\mathrm{Spec}(k(v))_{\acute{e}t}$ are constant and thus

$$H^{p+q}(\mathrm{Spec}(k(v))_{\acute{e}t}, \mathbb{Z}) = 0$$

Thus, it suffices to show $R^q (i_v)_* \mathbb{Z} = 0$ for $q > 0$ (by appealing to Leray SS). But $(i_v)_*$ is exact!

Indeed, for any $U \to X$ étale, we see that

$$(i_v)_* \mathcal{F}(U) = \mathcal{F}(U \times_X \mathrm{Spec}(k(v))) = \mathcal{F}^r \quad (\mathcal{F} \text{ is constant})$$

So for any SES of constant sheaves, we s

$$0 \to A \to B \to C \to 0$$

we see:

$$0 \to A^r \to B^r \to C^r \to 0$$

is exact.

$\square$

Ok, now we turn to an analysis of $H^*(X_{et}, g_* \mathbb{G}_{m,K})$.
We'll want to appeal to the Leray SS again, so really we want to study

$$H^*(\operatorname{Spec} K_{et}, \mathbb{G}_m)$$

But now K is a function field! So you know some wizardry must be involved. The only thing we can say for certain is

$$H^1(\operatorname{Spec} K_{et}, \mathbb{G}_m) = H^1(\operatorname{Spec} K_{Zar}, \mathbb{G}_m)$$
$$= 0$$

(There aren't many Zariski covers of Spec K ...)

thanks to Stefan.

To study H^2, we need to introduce the Brauer group. Once we've shown H^2 vanishes there's a cute trick from Galois cohomology that will get the rest to vanish.

<u>Central Simple Algebras:</u> A CSA over a field k is an associative algebra A s.t.

- A is <u>simple</u> (no non-trivial two-sided ideals)
- A is <u>central</u> ~ the center of A is exactly k.

Two CSAs A, B are said to be <u>similar</u> if $\exists n, m$ and an isomorphism of algebras

$$\operatorname{Mat}_{n\times n}(A) \cong \operatorname{Mat}_{m\times m}(B)$$

Similarity is an equiv relation, and the <u>Brauer group</u> of a field k is defined to be

$$\operatorname{Br}(k) := CSA^{fin}_{/k} / \sim$$

we require the algebras to be finite dim as vector spaces

Example: If k is algebraically closed then $Br(k) = 0$.

Proof: By Artin-Wedderburn, every Brauer class has a representative given by a division ring.

(leave as exercise?) For D a finite division ring over k, let $x \in D$. Then $k[x]$ is an integral extension of k, and thus a finite separable field extension. So $x \in k$ since k is alg. closed. $\square$

The key fact about Brauer groups that we'll need is that they give a geometric interpretation of H^2

Theorem: $Br(k) = H^2(\mathrm{Spec}\, k_{\acute{e}t}, \mathbb{G}_m)$

Proof: Omitted! $\square$

So if we want to show $H^2(\mathrm{Spec}\, K_{\acute{e}t}, \mathbb{G}_m) = 0$, we need to show the Brauer group vanishes.

The last step needed to reduce this to Tsen's Theorem is the construction of a "reduced norm" $N_A : A \to k$ for any CSA/k A.

For k algebraically closed, the norm is just going to be the determinant. If A is over a non-algebraically closed field k, fix an algebraic closure $\hat{k}$ and consider the composite

$$A \longrightarrow A \otimes_k \hat{k} \simeq \mathrm{Mat}_{n \times n}(\hat{k}) \xrightarrow{\det} \hat{k}$$

We know $A \otimes_k \hat{k} \simeq \mathrm{Mat}_{n \times n}(\hat{k})$ since $Br(\hat{k}) = 0$, and a theorem of Noether & Skolem tells us that any automorphism of $\mathrm{Mat}_{n \times n}(\hat{k})$ as a $\hat{k}$-algebra is inner - i.e. given by conjugation. In particular, the composite <u>doesn't depend</u> on the choice of identification $A \otimes_k \hat{k} \simeq \mathrm{Mat}_{n \times n}(\hat{k})$

Finally, let $G = Gal(\hat{k}/k)$ and note that G acts on $Mat_{n\times n}(\hat{k})$ by k-algebra automorphisms (not $\hat{k}$-algebra auts.)

Then $\det(g.A) = g.\det(A)$, so the map

$$A \to \hat{k}$$

factors through k, giving us our reduced norm $N_A : A \to k$.

To summarize, we've constructed for any A/k a map $N_A : A \to k$ s.t.

(i) If k is alg. closed, $N_A = \det$

(ii) N_A is compatible with base change

(iii) N_A is multiplicative.

So pick a basis $e_1, \dots, e_{n^2}$ of A. Then we can define a polynomial

$$p_A(x) = N_A(e_1 x_1 + \dots + e_{n^2} x_{n^2}) \in k[x_1, \dots, x_{n^2}]$$

s.t. p_A is homogeneous of degree n in n^2 variables.

If A is a division algebra, N_A takes A^* to k^*, and thus p_A has no nontrivial root!

<u>Definition:</u> A field k is said to be <u>quasi-algebraically closed</u> (or C_1) if for any homogeneous polynomial p of degree d in n variables with $d<n$ has a non-trivial root.

<u>Observation:</u> If k is C_1, then $Br(k)=0$ from the construction of the reduced norm.

Thus Tsen's theorem implies that $H^2(\operatorname{Spec} K_{et}, \mathbb{G}_m) = 0$!