ÉTALE COHOMOLOGY OF RINGS OF INTEGERS WITH COEFFICIENTS IN $\mathbb{G}_m$

EIVIND OTTO HJELLE

Let K be a number field and $\mathcal{O}_K$ the ring of integers in K. Following [Mil06, Chapter II.2] we will describe how to compute the étale cohomology groups

$$H^r(\operatorname{Spec}\mathcal{O}_K, \mathbb{G}_m),$$

or more generally $H^r(U, \mathbb{G}_m)$, where $U \subseteq \operatorname{Spec}\mathcal{O}_K$ is an open subscheme.

The computation is very similar to our computation of $H^r(X, \mathbb{G}_m)$ for X a complete nonsingular curve over an algebraically closed field, and will serve to review the techniques involved. In fact, our source treats both cases in parallel.

In Example 10 we will see that the groups $H^r(\operatorname{Spec}\mathbb{Z}, \mathbb{G}_m)$ are given by the following table.

r	0	1	2	3	≥ 4 even	≥ 5 odd
H^r	$\mathbb{Z}/2$	0	0	$\mathbb{Q}/\mathbb{Z}$	$\mathbb{Z}/2$	0

1. Tate cohomology

Let G be a finite group and M a G-module. Let $N_G = \sum_{g\in G} g$ and $I_G = \left\{\sum_{g\in G} n_g g : \sum_{g\in G} n_g = 0\right\}$. Define the Tate cohomology groups $\hat{H}^r(G, M)$ by

r	≤ -2	-1	0	≥ 1
$\hat{H}^r$	$H^{\lvert r\rvert-1}(G, M)$	$\frac{\operatorname{Ker} N_G}{I_G M}$	$\frac{M^G}{N_G M}$	$H^r(G, M)$

Example 1. Suppose G is the trivial group. Then $\hat{H}^r(G, M) = 0$ for all r.

Example 2. Let C_2 act on $\mathbb{C}^\times$ by complex conjugation. Then

$$\hat{H}^0(C_2, \mathbb{C}^\times) = \frac{\mathbb{R}^\times}{\mathbb{R}^\times_+} \cong \mathbb{Z}/2.$$

Note that we also have $\hat{H}^1(C_2, \mathbb{C}^\times) = H^1(\operatorname{Gal}(\mathbb{C}/\mathbb{R}), \mathbb{C}^\times) = 0$.

Theorem 3 ([Ser79, Section VIII.4]). *If G is cyclic, the groups $\hat{H}^r(G, M)$ only depend on the parity of r.*

2. Notation

For the remainder, let us fix some notation.

(1) K is a number field, for example $\mathbb{Q}$.
(2) $\mathcal{O}_K$ is the ring of integers in K, for example $\mathbb{Z}$ if $K = \mathbb{Q}$.
(3) $X = \operatorname{Spec}\mathcal{O}_K$.
(4) If v is a non-archimedean prime, then $K_v = \operatorname{Frac}\left(\mathcal{O}^h_{K,v}\right)$, where $\mathcal{O}^h_{K,v}$ is the henselization of $\mathcal{O}_{K,v}$.
(5) If v is a non-archimedean prime, $\kappa(v)$ is the residue fieldof K_v.
(6) If v is an archimedean prime, K_v is the completion of K with respect to v.
(7) U^0 is the set of (non-archimedean) places contained in U.
(8) $j\colon U \hookrightarrow X$ is the inclusion of an open subscheme.
(9) $g\colon \operatorname{Spec} K \hookrightarrow X$ is the inclusion of the generic point.

Convention. For an archimedean prime v and étale sheaf $\mathcal{F}$ on K_v, we set $H^r(K_v, \mathcal{F}) = \hat{H}^r(G_v, M_{\mathcal{F}})$, where $G_v = \mathrm{Gal}(K_v)$ and $M_{\mathcal{F}}$ is the G_v-module corresponding to $\mathcal{F}$.

Section 1 shows that if v is real then $H^r(K_v, \mathbb{G}_m)$ is $\mathbb{Z}/2$ for r even and 0 for r odd. And if v is complex then $H^r(K_v, \mathbb{G}_m) = 0$ for all r.

3. Cohomology computation

To compute $H^r(U, \mathbb{G}_m)$, we employ the divisor exact sequence

$$0 \to \mathbb{G}_m \to g_* \mathbb{G}_{m,K} \to \mathrm{Div}_U \to 0, \tag{1}$$

where $\mathrm{Div}_U = \bigoplus_{v \in U^0} i_{v,*}\mathbb{Z}$ is the sheaf of Weil divisors of U.

3.1. Cohomology of $g_*\mathbb{G}_{m,K}$.

We will use the following two theorems without proof.

Theorem 4 ([Sha72, p. 116]). *Let R be an excellent henselian DVR for which the residue field is algebraically closed. Then the fraction field of R is C_1*[1].

Theorem 5 ([Ser02, Section II.3]). *If L is a C_1 field then $H^r(L, \mathbb{G}_m) = 0$ for $r \geq 1$.*

Corollary 6. *The higher direct images $R^r g_* \mathbb{G}_{m,K} = 0$ for $r \geq 1$.*

Proof. Let $\overline{x} \to X$ be a geometric point. Then

$$(R^r g_* \mathbb{G}_{m,K})_{\overline{x}} \cong H^r(K_{\overline{x}}, \mathbb{G}_m),$$

where $K_{\overline{x}} = \mathrm{Frac}(\mathcal{O}_{K,\overline{x}})$ [Mil13, p. 81]. The ring $\mathcal{O}_{K,\overline{x}}$ is a strict henselization of $\mathcal{O}_{K,x}$ and satisfies the hypothesis of Theorem 4, so $K_{\overline{x}}$ is C_1. By Theorem 5 it follows that $(R^r g_* \mathbb{G}_{m,K})_{\overline{x}} = 0$ for $r \geq 1$. □

From the above we get that the Leray spectral sequence for g degenerates to isomorphisms

$$H^r(U, g_*\mathbb{G}_{m,K}) \cong H^r(K, \mathbb{G}_m).$$

Theorem 7 ([Mil06, Corollary I.4.21]). *For $r \geq 3$ we have isomorphisms*

$$H^r(K, \mathbb{G}_m) \cong \bigoplus_{v \text{ real}} H^r(K_v, \mathbb{G}_m) = \begin{cases} 0, & r \geq 3 \text{ odd}, \\ \bigoplus_{v \text{ real}} \mathbb{Z}/2, & r \geq 4 \text{ even}. \end{cases}$$

(Remember that $H^r(K_v, \mathbb{G}_m) = \hat{H}^r(K_v, \mathbb{G}_m)$.)

3.2. Cohomology of Div_U.

Since the points $v \in U^0$ are closed, $i_{v,*} \colon \mathrm{Spec}\,\kappa(v) \hookrightarrow U$ is exact by [Mil13, Corollary 8.4], so the Leray spectral sequence for i_v degenerates to isomorphisms $H^r(U, i_{v,*}\mathbb{Z}) \cong H^r(\kappa(v), \mathbb{Z})$. It follows that

$$H^r(U, \mathrm{Div}_U) \cong \bigoplus_{v \in U^0} H^r(U, i_{v,*}\mathbb{Z}) \cong \bigoplus_{v \in U^0} H^r(\kappa(v), \mathbb{Z}).$$

These groups are known:

Theorem 8 ([Mil06, Corollary II.1.2]). *The groups $H^r(\kappa(v), \mathbb{Z})$ are given by the following table.*

r	0	1	2	≥ 3
$H^r(\kappa(v), \mathbb{Z})$	$\mathbb{Z}$	0	$\mathrm{Br}(K_v)$	0

Remark 9. By [Mil06, Proposition I.A.1] the groups $\mathrm{Br}(K_v) \cong \mathbb{Q}/\mathbb{Z}$.

[1] Another word for C_1 is quasi-algebraically closed.

3.3. **The induced long exact sequence.** Using the above results, the long exact sequence for the divisor exact sequence (1) reads

$$\begin{array}{l}
0 \longrightarrow \Gamma(U, \mathcal{O}_U)^\times \longrightarrow K^\times \longrightarrow \bigoplus_{v \in U^0} \mathbb{Z} \\
\hookrightarrow \operatorname{Pic}(U) \longrightarrow 0 \longrightarrow 0 \\
\hookrightarrow H^2(U, \mathbb{G}_m) \longrightarrow \operatorname{Br}(K) \longrightarrow \bigoplus_{v \in U^0} \operatorname{Br}(K_v) \\
\hookrightarrow H^3(U, \mathbb{G}_m) \longrightarrow 0
\end{array}$$

and for $r \geq 4$ we have isomorphisms

$$H^r(U, \mathbb{G}_m) \cong H^r(K, \mathbb{G}_m) \cong \bigoplus_{v \text{ real}} H^r(K_v, \mathbb{G}_m) \cong \begin{cases} \bigoplus_{v \text{ real}} \mathbb{Z}/2, & r \text{ even}, \\ 0, & r \text{ odd}. \end{cases}$$

3.4. **Interlude on Brauer groups.** Global class field theory yields an exact sequence

$$0 \to \operatorname{Br}(K) \to \bigoplus_{\text{all } v} \operatorname{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0, \tag{2}$$

where the sum is taken over all places v of K [Mil06, Theorem I.A.7]. We can describe the maps $\operatorname{Br}(K_v) \to \mathbb{Q}/\mathbb{Z}$:

(1) If v is non-archimedean, it is an isomorphism $\operatorname{Br}(K_v) \cong \mathbb{Q}/\mathbb{Z}$ [Mil06, Proposition I.A.1].
(2) If v is real, it maps $\operatorname{Br}(K_v) \cong \operatorname{Br}(\mathbb{R}) \cong \mathbb{Z}/2$ onto $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$.
(3) If v is complex, $\operatorname{Br}(K_v) = 0$ and there is nothing to say.

3.5. **Another exact sequence.** Consider the "snake lemma diagram"

$$\begin{array}{ccccccccc}
 & & H^2(U, \mathbb{G}_m) & \longrightarrow & \bigoplus_{v \notin U} \operatorname{Br}(K_v) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Br}(K) & \longrightarrow & \bigoplus_{\text{all } v} \operatorname{Br}(K_v) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_{v \in U^0} \operatorname{Br}(K_v) & = & \bigoplus_{v \in U^0} \operatorname{Br}(K_v) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^3(U, \mathbb{G}_m) & \longrightarrow & 0 & & & &
\end{array}$$

The second row is (2), so it is exact. In the first column the top and bottom element is the kernel and cokernel of the middle map by section 3.3. By the snake lemma it follows that there is a map $\mathbb{Q}/\mathbb{Z} \to H^3(U, \mathbb{G}_m)$ such that

$$0 \to H^2(U, \mathbb{G}_m) \to \bigoplus_{v \notin U} \operatorname{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to H^3(U, \mathbb{G}_m) \to 0 \tag{3}$$

is exact. Note that the sum is taken over all places v not in U, including archimedean ones.

3.6. **Examples.**

Example 10 ($K = \mathbb{Q}$ and $U = \operatorname{Spec} \mathbb{Z}$)**.** Let H^r be short hand for $H^r(\operatorname{Spec} \mathbb{Z}, \mathbb{G}_m)$.

By definition $H^0 = \mathbb{Z}^\times = \{\pm 1\} \cong \mathbb{Z}/2$.

The group $H^1 = \operatorname{Pic}(\mathbb{Z})$ is the ideal class group of $\mathbb{Z}$, which is 0 because $\mathbb{Z}$ is a PID.

The sequence (3) reads
$$0 \to H^2 \to \mathbb{Z}/2 \to \mathbb{Q}/\mathbb{Z} \to H^3 \to 0,$$
where the second map is an isomorphism onto $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$. It follows that $H^2 = 0$, and $H^3 \cong \mathbb{Q}/\frac{1}{2}\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}$.

From section 3.3 we see that for $r \geq 4$ the group $H^r \cong \mathbb{Z}/2$ if r is even and $H^r \cong 0$ if r is odd.

Example 11 (K has no real primes and $U = \operatorname{Spec} \mathcal{O}_K$)**.** Assume that K has no real primes. If $K = \mathbb{Q}(\gamma)$, this is equivalent to requiring that the minimal polynomial of γ has no real roots. For example, we could take $K = \mathbb{Q}(i)$. For $H^r = H^r(\operatorname{Spec} \mathcal{O}_K, \mathbb{G}_m)$, we have the following table.

r	0	1	2	3	≥ 4
H^r	$\mathcal{O}_K^\times$	$\operatorname{Pic}(\mathcal{O}_K)$	0	$\mathbb{Q}/\mathbb{Z}$	0

References

[Mil06] J.S. Milne. *Arithmetic Duality Theorems*. Second. BookSurge, LLC, 2006, pp. viii+339. ISBN: 1-4196-4274-X.

[Mil13] James S. Milne. *Lectures on Etale Cohomology (v2.21)*. Available at www.jmilne.org/math/. 2013.

[Ser02] Jean-Pierre Serre. *Galois cohomology*. English. Springer Monographs in Mathematics. Translated from the French by Patrick Ion and revised by the author. Springer-Verlag, Berlin, 2002, pp. x+210. ISBN: 3-540-42192-0.

[Ser79] Jean-Pierre Serre. *Local fields*. Vol. 67. Graduate Texts in Mathematics. Translated from the French by Marvin Jay Greenberg. Springer-Verlag, New York-Berlin, 1979, pp. viii+241. ISBN: 0-387-90424-7.

[Sha72] Stephen S. Shatz. *Profinite groups, arithmetic, and geometry*. Annals of Mathematics Studies, No. 67. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1972, pp. x+252.