IE 383 –Service Operations Management

FINAL EXAM – SOLUTIONS

Grading:

Problem	Max Points	Degree of difficulty	Score received
1	25	Relatively easy (computational workforce scheduling)	
2	30	Relatively easy (formulation location)	
3	45	Moderate (computational queueing)	
TOTAL	100		

I suggest you work on the two easy problems quickly and leave time for the moderate problem.

Glance at the entire exam to see what you can do quickly and easily.

Don't panic at 10 pages! The amount of work should be doable. There is a lot of white space.

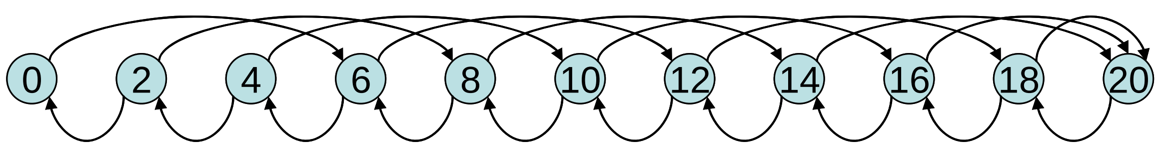
Problem 1:

Consider the problem of finding the optimal starting times for employees. Each employee works a 6 hour shift with no breaks. Each period below is 2 hours long. In other words, we need 4 people on duty during the first 2 hours, 6 during hours 2 and 3, 8 during hours 4 and 5, and so on.

PERIODS

Period #	Begin	End	Needed
1	0	2	4
2	2	4	6
3	4	6	8
4	6	8	5
5	8	10	10
6	10	12	3
7	12	14	9
8	14	16	10
9	16	18	7
10	18	20	5

a) Draw the diagram that corresponds to the network model for minimizing the number of employees needed to ensure that each period has the required number of people. Clearly show the nodes and links. For each link, include a triplet showing (lower bound, unit cost, upper bound).



The lower bound on the upper links is 0, the unit cost is 1 and the upper bound is infinity.

The lower bound on the lower links is the number of people required to be on duty during the given interval, the unit cost is 0 and the upper bound is infinity.

b) A naïve approach would be as follows: look at the first 3 periods (since employees work for 3 periods). Find the number needed to start at the beginning of period 1 to adequately serve these 3 periods (e.g., 8 above). Repeat for periods 4, 5, and 6; then for periods 7, 8, and 9; and finally for period 10 alone. Use this algorithm to solve the problem heuristically. **How many employees do you need?**

You would use 8, 10, 10, and then 5 employees or a total of 33.

c) It turns out that a greedy algorithm can solve this problem optimally. The way this works is to find the minimum number of people needed to start at the beginning of the first period needed to ensure that the first period is served adequately. In this case, this is clearly 4. These people will then serve other subsequent periods. Subtract that number (4) from the number required in all periods during which those people work (periods 1, 2, and 3). Find the next period that needs people, and continue in the same way. Use this algorithm to find the minimum number of people needed to ensure that you have the minimum number on duty at all times. **Complete the table to the right showing how many people start in each period and how many total employees you need.**

PERIODS

Period #	Begin	End	Needed	# to Start
1	0	2	4	4
2	2	4	6	2
3	4	6	8	2
4	6	8	5	1
5	8	10	10	7
6	10	12	3	0
7	12	14	9	2
8	14	16	10	8
9	16	18	7	0
10	18	20	5	0

TOTAL				26

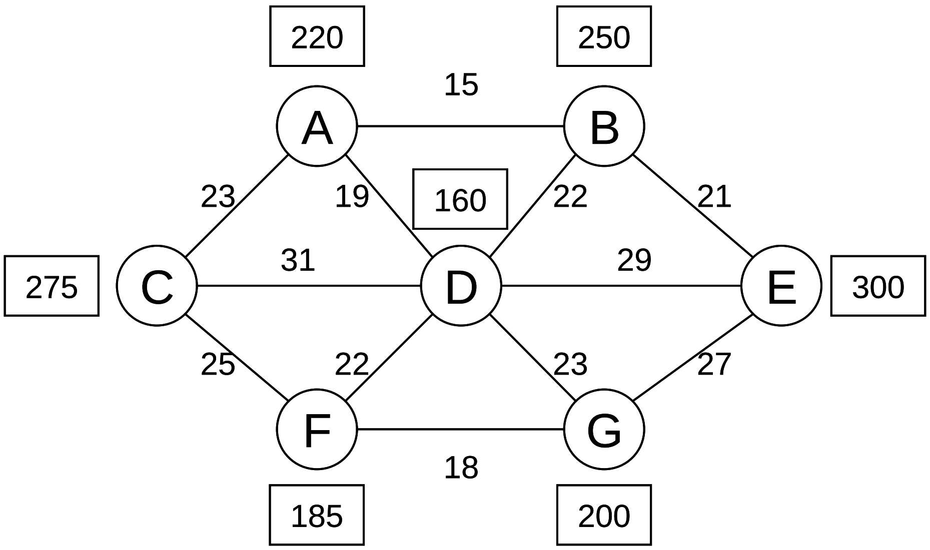
d) By what percentage (of the number of required employees) does the solution using the naïve approach exceed the optimal solution value?

The percentage is (33-26)/26 or 7/26 or 27%.

Problem 2:

Consider the following network:

Numbers shown in squares beside the nodes are the demands per week at the node.



Consider the hierarchical objective problem of **FIRST** finding the minimum number of facilities needed to cover all nodes within a coverage distance of 23, and then, from among all of the alternate optima for this problem, **SECOND** finding the solution that maximizes the number of **demands** that are covered two or more times.

Define the following notation:

Inputs:

J	set of demand nodes
K	set of candidate nodes (same as the set of demand nodes in this case)
a_{jk}	1 if demand node j can be covered by a facility located at node k; 0 if not
h_j	demand at node j
W	weight on the primary objective of minimizing the number of facilities

Decision variables:

X_k	1 if a facility is located at candidate site k; 0 if not
Z_j	1 if demand node j is covered 2 or more times; 0 if not

a) Formulate the hierarchical objective of FIRST minimizing the number of facilities needed and SECOND maximizing the number of demands that are covered 2 or more times. *Note that this should be a **single** function. This should be formulated in terms of **notation**.*

Minimize $W \sum_{k \in K} X_k - \sum_{j \in J} h_j Z_j$

b) Formulate the constraints for this model including any integrality constraints. Explain the constraints using notation as well as words.

Subject to: $\sum_{k \in K} a_{jk} X_k - Z_j \geq 1 \qquad \forall j \in J$

$X_k \in \{0,1\} \qquad \forall k \in K$

$Z_j \in \{0,1\} \qquad \forall j \in J$

c) Do the Z_j variables have to be constrained to be binary, or can you simply use a constraint of the following form: $0 \le Z_j \le 1$ for all j? Briefly justify your answer.

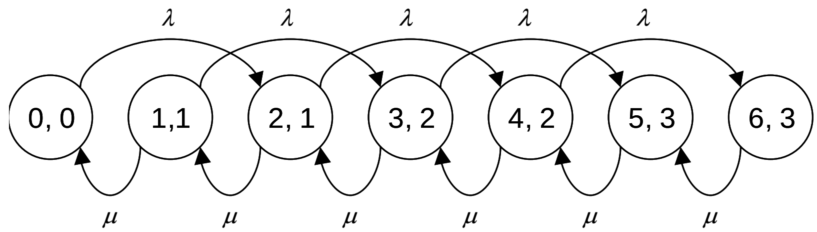
No, you do not. Once you require the location variables to be integer, the quantity $\sum_{k \in K} a_{jk} X_k$ will be integer and then the value of Z_j will be 0 if $\sum_{k \in K} a_{jk} X_k = 1$ and 1 if $\sum_{k \in K} a_{jk} X_k > 1$ since all demands are positive.

d) How large should W be to ensure that the combined objective function of part (a) first minimizes the number of facilities and then selects the solution that maximizes the number of multiply covered demands from among the alternate optima? Briefly justify your answer. What you want is the smallest possible value of W.

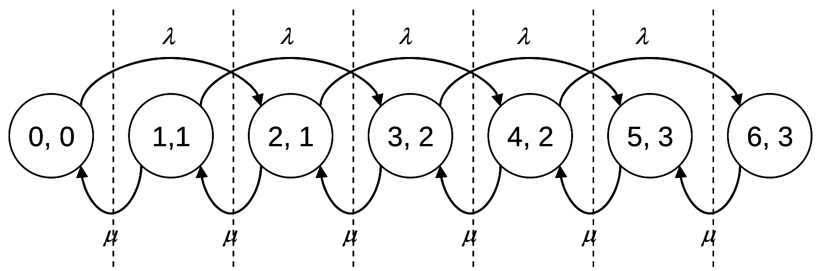
W should be greater than the sum of the demands so that adding an extra facility will hurt the first term of the objective function more than it can possibly benefit the second term.

Problem 3:

Consider again the M/E$_2$/1 queue with a finite capacity. In the case shown below, the system can accommodate a total of 3 customers. Again, as in the problem set, the dual notation on the state space can be interpreted as follows. The first number of gives the number of exponentially distributed work units that are in the system, and the second gives the number of customers in the system.



If we now think about the steady state balance equations in terms of "the rate to the right equals the rate to the left", we can envision the following diagram:



For the cut between states (0,0) and (1,1) we would write the equation, $\lambda P_{0,0} = \mu P_{1,1}$ while the equation for the cut between states (1,1) and (2,1) would be $\lambda(P_{0,0} + P_{1,1}) = \mu P_{2,1}$.

a) Write down all 6 such equations. Be careful with the last equation!

$$\lambda P_{0,0} = \mu P_{1,1}$$
$$\lambda(P_{0,0} + P_{1,1}) = \mu P_{2,1}$$
$$\lambda(P_{1,1} + P_{2,1}) = \mu P_{3,2}$$
$$\lambda(P_{2,1} + P_{3,2}) = \mu P_{4,2}$$
$$\lambda(P_{3,2} + P_{4,2}) = \mu P_{5,3}$$
$$\lambda(P_{4,2}) = \mu P_{6,3}$$

b) Assume that $\lambda = 1$ and $\mu = 1$, meaning that the mean service time is 2 or k/μ. Solve the equations in part (a) terms of $P_{0,0}$.

$\lambda P_{0,0}/\mu = P_{1,1}$
$\lambda(P_{0,0} + P_{1,1})/\mu = P_{2,1}$
$\lambda(P_{1,1} + P_{2,1})/\mu = P_{3,2}$
$\lambda(P_{2,1} + P_{3,2})/\mu = P_{4,2}$
$\lambda(P_{3,2} + P_{4,2})/\mu = P_{5,3}$
$\lambda(P_{4,2})/\mu = P_{6,3}$

Let $P_{0,0} = 1$, then $P_{1,1} = 1$, $P_{2,1} = 2$, $P_{3,2} = 3$, $P_{4,2} = 5$, $P_{5,3} = 8$, $P_{6,3} = 5$

Note that the total is 25.

c) Using the fact that the sum of the 7 probabilities must be equal to 1, find the state probabilities.

$P_{0,0} = 0.04$, $P_{1,1} = 0.04$, $P_{2,1} = 0.08$, $P_{3,2} = 0.12$, $P_{4,2} = 0.2$, $P_{5,3} = 0.32$, $P_{6,3} = 0.2$

d) Find the expected number of customers in the system, L.

$$L = 0.04 + 0.08 + 2(0.12 + 0.20) + 3(0.32 + 0.20) = 0.12 + 0.64 + 1.56 = 2.32$$

e) Find the effective arrival rate, λ_{eff}.

$$\lambda_{eff} = 1(1 - P_{5,3} - P_{6,3}) = 0.48$$

f) Find the expected time in the system, W.

$$W = L/\lambda_{eff} = 2.32/0.48 = 4.8333$$

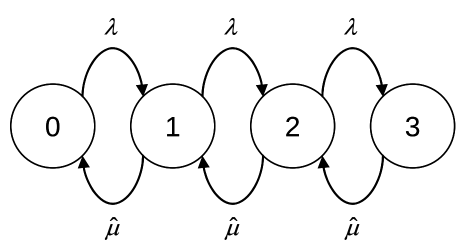
g) Find the expected waiting time in the queue before service, W_q.

$$W_q = W - E(\text{service time}) = 4.8333 - 2 = 2.8333$$

h) Find the expected number in the queue waiting for service, L_q.

$$L_q = \lambda_{eff} W_q = 0.48(2.8333) = 1.36$$

i) Now consider the M/M/1 queue with a finite capacity of 3, whose state space is shown to the right. In this figure, I am using $\hat{\mu}$ to distinguish this service rate from the value of μ that we used earlier. In particular, if the mean service time is to be the same for this queue and for the M/E₂/1 queue above, we would have $\hat{\mu} = 0.5$. For this value of $\hat{\mu}$ and $\lambda = 1$, find the state probabilities.



$$\begin{aligned} \lambda P_0 &= \mu P_1 \\ \lambda P_1 &= \mu P_2 \\ \lambda P_2 &= \mu P_3 \end{aligned} \quad \text{so} \quad \begin{aligned} P_1 &= 2P_0 \\ P_2 &= 2P_1 = 4P_0 \\ P_3 &= 2P_2 = 8P_0 \end{aligned} \quad \text{so} \quad \begin{aligned} P_0 &= 1/15 \\ P_0 &= 2/15 \\ P_0 &= 4/15 \\ P_0 &= 8/15 \end{aligned}$$

j) Find the average number in the M/M/1 finite queue of part (i), the effective arrival rate, the average time in the system, the average time in the queue, and the average number in the queue. Fill in the table below:

	L	λ_{eff}	W	W_q	L_q
$M/E_2/1$	**2.320**	**0.480**	**4.833**	**2.833**	**1.360**
M/M/1	**2.267**	**0.467**	**4.857**	**2.857**	**1.333**
Difference	**0.053**	**0.013**	**-0.024**	**-0.024**	**0.027**

k) **Briefly justify** why the relationship between the time in the system for the M/E2/1 queue and the corresponding time for the M/M/1 queue should be what it is.

Yes, this makes sense since there is more variability in the service time distribution in the exponential distribution and so there should be better overall performance from the $M/E_2/1$ queue in terms of time in the system and effective arrival rate.