IE 383 –Service Operations Management **Spring, 2006**
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QUIZ

Problem 1:

You are in charge of assigning classes to rooms and times at a small community college that offers evening classes from 6 to 10 in the evening. Some classes are scheduled MW from 6-8 and 8-10, while others are TTh from 6-8 and 8-10. In other words, there are 4 different time slots as shown below. Also shown in the table are three different rooms that you have available to you for use in the evenings. The small room (110) has a capacity of 25 students. Room 121 can hold 45 and room 135 can hold 60. Each room/time combination has been assigned a number for easier reference below.

Days	Time	Room	Capacity	Room/Time Combination Number
M W	6-8	110	25	1
		121	45	2
		135	60	3
	8-10	110	25	4
		121	45	5
		135	60	6
T Th	6-8	110	25	7
		121	45	8
		135	60	9
	8-10	110	25	10
		121	45	11
		135	60	12

There are therefore 12 room/time combinations. In addition, you have enrollment information for the 11 courses that you will be offering. This information is given in the table to the right. Each course has an ID number for ease of reference below as well.

Course	Enrollment	Course ID #
Accounting	27	1
Advanced Excel	16	2
Advanced Word	24	3
Art Appreciation	52	4
Current Events	43	5
English Basics	55	6
Excel Basics	38	7
Nutrition	56	8
Personal Finance	42	9
US Politics	55	10
Word Basics	41	11

Clearly you do not want to assign a large enrollment class (e.g., US Politics with an enrollment of 55) to a small classroom (e.g., room 110 or 121). If you do, you incur a penalty that is 100 penalty units for each student above and beyond the capacity of the room; i.e., for each student you have to notify that he/she cannot take the course.

While it is less serious, you also do not want to assign small classes (e.g., Advanced Excel) to a room with a lot of excess capacity. (I once had to teach IE 303 in the main Tech Auditorium and it was terrible for everyone – students and faculty!) There is a penalty of 20 for each excess space in a room.

Let us define the following notation:

Inputs:

J **set of classes indexed by *j***
K **set of room/time combinations**

c_u **Cost of undercapcity per student (100 penalty units)**
c_o **Cost of overcapacity per student (20 penalty units)**

h_j **enrollment in class *j***
s_k **space available in room/time *k***
p_{jk} **penalty for assigning class *j* to room/time *k***

Decision variables:

Y_{jk} **1 if class *j* is assigned to room/time *k***

a) Using the notation defined above, write down a formula or expression for the penalty p_{jk} in terms of the costs of undercapacity c_u and overcapacity c_o as well as the enrollment in the course h_j and the room capacity, s_k.

$$p_{jk} = \begin{cases} c_u\left(h_j - s_k\right) & h_j - s_k \geq 0 \\ c_o\left(s_k - h_j\right) & h_j - s_k < 0 \end{cases}$$

b) Using the notation defined above, write down the objective of minimizing the total penalty of all assignments.

Minimize $$\sum_{k \in K} \sum_{j \in J} p_{jk} Y_{jk}$$

c) Using the notation defined above, write down the constraint that the number of classes assigned to a room/time combination cannot exceed 1.

$$\sum_{j \in J} Y_{jk} \leq 1 \qquad \forall k \in K$$

d) Using the notation defined above, write down the constraint that each class has to be assigned to a room/time.

$$\sum_{k \in K} Y_{jk} = 1 \qquad \forall j \in J$$

e) With this formulation, do the variables Y_{jk} need to be binary, or can they simply be non-negative. **Briefly justify your answer.**

No, they do not since this is basically an assignment or transportation problem with all integer right-hand side values. Thus, the solution will be all integer and will automatically be 0/1 in this case.

f) The solution to the problem is given below (on the next page): The objective function (total penalty) is 1160. If you stare at this for some time, you will realize that the two EXCEL classes are scheduled on different days (Excel Basics on MW and Advanced Excel on T Th). The same is true for the two WORD classes (Word Basics on MW and Advanced Word on T Th). The two Excel classes have the same instructor (Ms. Workbook) and the two Word classes have the same instructor (Mr. Paragraph). Both Ms. Workbook and Mr. Paragraph have said they want to teach on one night (either MW or T Th). (They do not have to be on the same night, but both of the Excel classes have to be on either MW or T Th and similarly both of the Word classes have to be on MW or T Th). **Formulate linear constraints that will ensure that the schedule meets these requirements.**

$$\sum_{k=1}^{6} Y_{2,k} + \sum_{k=7}^{12} Y_{7,k} \le 1 \qquad \sum_{k=1}^{6} Y_{2,k} = \sum_{k=1}^{6} Y_{7,k}$$

$$\sum_{k=7}^{12} Y_{2,k} + \sum_{k=1}^{6} Y_{7,k} \le 1 \qquad \sum_{k=7}^{12} Y_{2,k} = \sum_{k=7}^{12} Y_{7,k}$$

or you could use

$$\sum_{k=1}^{6} Y_{3,k} + \sum_{k=7}^{12} Y_{11,k} \le 1 \qquad \sum_{k=1}^{6} Y_{3,k} = \sum_{k=1}^{6} Y_{11,k}$$

$$\sum_{k=7}^{12} Y_{3,k} + \sum_{k=1}^{6} Y_{11,k} \le 1 \qquad \sum_{k=7}^{12} Y_{3,k} = \sum_{k=7}^{12} Y_{11,k}$$

Note that these allow the teachers to teach the same class at the same time. To preclude this, we can use the following:

$$\sum_{k=1}^{3} Y_{2,k} = \sum_{k=4}^{6} Y_{7,k} \text{ and } \sum_{k=4}^{6} Y_{2,k} = \sum_{k=1}^{3} Y_{7,k}$$

$$\sum_{k=7}^{9} Y_{2,k} = \sum_{k=10}^{12} Y_{7,k} \text{ and } \sum_{k=10}^{12} Y_{2,k} = \sum_{k=7}^{9} Y_{7,k}$$

$$\sum_{k=1}^{3} Y_{3,k} = \sum_{k=4}^{6} Y_{11,k} \text{ and } \sum_{k=4}^{6} Y_{3,k} = \sum_{k=1}^{3} Y_{11,k}$$

$$\sum_{k=7}^{9} Y_{3,k} = \sum_{k=10}^{12} Y_{11,k} \text{ and } \sum_{k=10}^{12} Y_{3,k} = \sum_{k=7}^{9} Y_{11,k}$$

		Days	M W	M W	M W	M W	M W	M W	T Th	T Th	T Th	T Th	T Th	T Th
		Time	6-8	6-9	6-10	8-10	8-11	8-12	6-8	6-9	6-10	8-10	8-11	8-12
		Room	110	121	135	110	121	135	110	121	135	110	121	135
		Room/Time Combination Number	1	2	3	4	5	6	7	8	9	10	11	12
Course ID #	Course	Enrollment												
1	Accounting	27				**1**								
2	Advanced Excel	16							**1**					
3	Advanced Word	24										**1**		
4	Art Appreciation	52						**1**						
5	Current Events	43											**1**	
6	English Basics	55												**1**
7	Excel Basics	38					**1**							
8	Nutrition	56									**1**			
9	Personal Finance	42								**1**				
10	US Politics	55			**1**									
11	Word Basics	41		**1**										

g) With these additional requirements, will the new total penalty be less than the old penalty, the same as the old penalty, or greater than the old penalty (or maybe something else). **Again, briefly justify your answer.**

The objective function will either stay the same or go up since you are adding a constraint to a minimization problem.

Problem 2:

Consider the P-center problem that we discussed in class of minimizing the maximum over all nodes distance between a demand node and the facility to which it is assigned. We define the following notation:

Inputs:

J **set of demand nodes, indexed by *j***
K **set of candidate facility locations, indexed by *k***

h_j **demand at node *j***
d_{jk} **distance between demand node *j* and candidate site *k***
P **number of facilities to locate**

Decision variables:

X_k **1 if we locate a facility at candidate site *k*; 0 if not**
Y_{jk} **1 if demand node *j* is assigned to a facility at candidate site *k***
Q **maximum assigned distance (the quantity we want to minimize)**

The objective of the problem is going to be

Minimize Q

a) Using the notation above, formulate the constraint that each demand node is assigned.

$$\sum_{k \in K} Y_{jk} = 1 \qquad \forall j \in J$$

b) Formulate the constraint that says that demand nodes can only be assigned to open facilities (sites that have been selected)

$$Y_{jk} \leq X_k \qquad \forall j \in J; \forall k \in K$$

c) Formulate the constraint that says that you locate exactly *P* facilities.

$$\sum_{k \in K} X_k = P$$

d) What other constraints do you need? For each additional constraint, state the constraint in **words** as well as **mathematical notation**.

Definition of the maximum distance in terms of the assignment variables

$$\sum_{k \in K} d_{jk} Y_{jk} \le Q \qquad \forall j \in J$$

Integrality

$$X_k \in \{0,1\} \qquad \forall k \in K$$

$$Y_{jk} \in \{0,1\} \qquad \forall j \in J; \forall k \in K$$

e) The objective for the P-median problem is simply to minimize the demand-weighted **average** distance which can be written as: $\left(\sum_{k \in K} \sum_{j \in J} h_j d_{jk} Y_{jk} \right) \Big/ \left(\sum_{j \in J} h_j \right)$. The constraints for this problem are the same as the constraints in parts (a, b, c and maybe d) above. Therefore, we can use a combined model to find the tradeoff between the two objectives (minimizing the average and the maximum distances). Suppose we have two solutions as shown in the table to the right. If you want to use the weighting method to find a new solution on the tradeoff curve, what should the value of W be if the weighted objective is $W(\text{Avg Dist}) + (1-W)(\text{Max Dist})$? **Show your work.**

	Distances	
Solution	**Average**	**Max**
1	150	500
2	200	425

$$W150 + (1-W)500 = W200 + (1-W)425$$

$$(1-W)75 = 50W$$

$$75 = 125W$$

$$W = 75/125 = 0.6$$

Problem 3:

Consider the **simplified** problem faced by the admissions office of a medium-sized liberal arts college. The college wants to have a freshman class of 2000. From previous experience, they know that 40% of the students they admit will actually decide to come to the college.

If the number that decide to come exceeds 2000, there is an additional cost to the college in terms of housing the students for the fall quarter in expensive housing. This amounts to \$3000 per student. On the other hand, if fewer than 2000 decide to accept the offer of admission, the college loses tuition revenue. This cost is \$8000 per student.

a) If the college admits N students, what is mean and variance of the number of students who actually decide to go to the college?

$$\text{mean} = 0.4N$$
$$\text{variance} = 0.24N$$
from the binomial distribution

b) What is the approximate distribution of the number of students who will enroll at the college as a function of N?

It is approximately NORMAL with the mean and variance given above.

For the rest of the problem, you may find the following useful:

$$\int_{-\infty}^{Q} x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) dx = \int_{-\infty}^{Q} x\varphi\left(\frac{x-\mu}{\sigma}\right) dx$$
$$= \mu\Phi\left(\frac{Q-\mu}{\sigma}\right) - \sigma\varphi\left(\frac{Q-\mu}{\sigma}\right)$$

where

$\varphi(x) =$ standard Normal density function evaluated at x

$\Phi(\mathrm{x}) =$ cumulative standard Normal density function evaluated at x

c) By what expected number of students will the college be short of its target class of 2000? *Hint: Think about this in two parts: (1) if the number who decide to come to the college is less than or equal to 2000 and (2) if the number who decide to come is more than 2000.*

$$2000-\int_{-\infty}^{2000} x\phi(x)dx-2000\int_{2000}^{\infty}\phi(x)dx=2000-\left\{0.4N\Phi(y)-\sqrt{0.24N}\varphi(y)\right\}-2000\{1-\Phi(y)\}$$

$$=(2000-0.4N)\Phi(y)+\sqrt{0.24N}\varphi(y)$$

where

$$y=\frac{2000-0.4N}{\sqrt{0.24N}}$$

and

$$\phi(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

$$=\frac{1}{\sqrt{2\pi}\sqrt{0.24N}}\exp\left\{\frac{-1}{2}\left(\frac{x-0.4N}{\sqrt{0.24N}}\right)^2\right\}$$

d) What is the expected number of students that the college will have to house in expensive housing as a function of N? (i.e., what is the expected amount by which the number of students enrolling at the college exceeds 2000?)

$$\int_{2000}^{\infty}(x-2000)\phi(x)dx=\int_{2000}^{\infty}x\phi(x)dx-2000\int_{2000}^{\infty}\phi(x)dx$$

$$=0.4N-\int_{-\infty}^{2000}x\phi(x)dx-2000\int_{2000}^{\infty}\phi(x)dx$$

$$=0.4N-\left\{0.4N\Phi\{y\}-\sqrt{0.24N}\varphi\{y\}\right\}-2000\{1-\Phi\{y\}\}$$

$$=(0.4N-2000)\{1-\Phi\{y\}\}+\sqrt{0.24N}\varphi\{y\}$$

e) Find the optimal number of students to admit to minimize the expected cost – the cost of having too few students come (at \$8000 per student) and the cost of having too many come (at \$3000 per student).

We need to solve:

$$P(\text{too few})8000 = P(\text{too many})3000$$

$$\Phi\left(\frac{2000-0.4N}{\sqrt{0.24N}}\right)8000 = \left[1-\Phi\left(\frac{2000-0.4N}{\sqrt{0.24N}}\right)\right]3000$$

$$\Phi\left(\frac{2000-0.4N}{\sqrt{0.24N}}\right)11000 = 3000$$

$$\Phi\left(\frac{2000-0.4N}{\sqrt{0.24N}}\right) = \frac{3}{11}$$

or

$$\Phi\left(\frac{3}{11}\right)^{-1} = \frac{2000-0.4N}{\sqrt{0.24N}}$$

$$\frac{2000-0.4N}{\sqrt{0.24N}} = -0.605$$

$$N \approx 5052$$