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IE 310 Operations Research **Winter, 2003**
Mark S. Daskin **Dept. of IE/MS** **Northwestern University**

Introduction to Operations Research

FINAL EXAM – SOLUTIONS

Problem 1 (30 percent): (Probability and Queuing Theory)

The Erlang-k distribution is the sum of *k* independent and identically distributed exponential random variables, each with parameter $k\mu$. In other words, if X_i is the ith exponentially distributed random variable and E_k is the Erlang-k random variable, then $E_k = X_1 + X_2 + \bullet\bullet\bullet + X_k$.

a) What is the mean and variance of any one of the exponential distributions that make up the Erlang-k distribution? *Note that both the mean and the variance should depend on the parameter k of the Erlang-k distribution.*

$$\boxed{\text{Mean} = \frac{1}{ku} \qquad \text{Variance} = \frac{1}{k^2\mu^2}}$$

b) What is the mean and variance of the Erlang-k distribution? *Recall that the mean of the sum is the sum of the means and the variance of the sum (of independent random variables) is the sum of the variances.*

$$\boxed{\text{Mean} = \frac{1}{u} \qquad \text{Variance} = \frac{1}{k\mu^2}}$$

c) What is the coefficient of variation (*standard deviation divided by the mean*) of the Erlang-k distribution?

$$\boxed{\text{Coef of variation} = \frac{\sqrt{\frac{1}{k\mu^2}}}{\frac{1}{\mu}} = \frac{1}{\sqrt{k}}}$$

d) Suppose the service time distribution in a single-server queue has an Erlang-k distribution. Find the average number of people in the system assuming the arrivals follow a Poisson process.

$$\boxed{L = \rho + \frac{\rho^2 + \lambda^2\sigma^2}{2(1-\rho)} = \rho + \frac{\rho^2 + \rho^2/k}{2(1-\rho)} = \rho + \frac{\rho^2\left(1+\frac{1}{k}\right)}{2(1-\rho)}}$$

e) What is the average time in the system (W), the average time in the queue (W_q) and the average number in the queue (L_q)?

$$W = \frac{L}{\lambda} = \frac{1}{\mu} + \frac{\rho\left(1+\frac{1}{k}\right)}{2\mu(1-\rho)}$$

$$W_q = W - \frac{1}{\mu} = \frac{\rho\left(1+\frac{1}{k}\right)}{2\mu(1-\rho)}$$

$$L_q = \lambda W_q = \frac{\rho^2\left(1+\frac{1}{k}\right)}{2(1-\rho)}$$

f) If $k=1$, show that you get the result for the M/M/1 queue (e.g., $L_q = \frac{\rho^2}{1-\rho}$). Show that your result from part (e) reduces to this value for the M/M/1 queue.

$$\boxed{L_q = \lambda W_q = \frac{\rho^2\left(1+\frac{1}{k}\right)}{2(1-\rho)} = \frac{\rho^2 2}{2(1-\rho)} = \frac{\rho^2}{(1-\rho)}}$$

g) As $k \to \infty$, show that you get the result for the M/D/1 queue (e.g., $L_q = \frac{\rho^2}{2(1-\rho)}$). Show that your result from part (e) reduces to this value for the M/D/1 queue.

$$\boxed{\lim_{k\to\infty} L_q = \lim_{k\to\infty} \lambda W_q = \lim_{k\to\infty} \frac{\rho^2\left(1+\frac{1}{k}\right)}{2(1-\rho)} = \frac{\rho^2}{2(1-\rho)}}$$

Problem 2 (25 percent): (Problem Formulation)

In class we considered a simple transportation problem in which goods flow from a producer to a customer. In this problem, we want to consider an extension of that problem in which there is a single plant with a known location, multiple **candidate** locations for distribution centers, and customers with known locations and demands. In particular, we will define the following notation:

Inputs and Sets:

$\mathbf{I}$	set of customer locations
$\mathbf{J}$	set of candidate DC locations
h_i	demand per year at customer i
c_{ij}	unit cost of shipping one item from candidate DC site j to customer i
b_j	unit cost of shipping one unit from the plant to candidate DC site j
p	unit cost of producing an item
f_j	fixed cost (annualized) of locating a DC at candidate site j

Decision variables:

$$X_j = \begin{cases} 1 & \text{if we locate a DC at candidate site j} \\ 0 & \text{if not} \end{cases}$$

W_{ij} = number of units shipped from DC j to customer i per year

V_j = number of units shipped from the plant to DC j

a) With this notation, formulate the objective function which should consist of minimizing the annual cost of (1) producing the items, (2) locating DCs, (3) shipping from the plant to the DCs, and (4) shipping from the DCs to the customers. *Be sure you show all 4 terms and label each one.*

producing the items	$p\sum_{j\in\mathbf{J}} V_j +$
locating DCs	$\sum_{j\in\mathbf{J}} f_j X_j +$
shipping from the plant to the DCs	$\sum_{j\in\mathbf{J}} b_j V_j +$

shipping from the DCs to the customers	$\sum_{i\in\mathbf{I}}\sum_{j\in\mathbf{J}} c_{ij} W_{ij}$

b) Now consider the constraint that says that the amount shipped into each distribution center must equal the amount shipped out of the DC. Formulate this constraint in terms of the notation above.

$$\boxed{V_j = \sum_{i\in\mathbf{I}} W_{ij} \qquad \forall j \in \mathbf{J}}$$

c) Now consider the constraint that states that the demand must be satisfied at every demand node. Formulate this constraint in terms of the notation above.

$$\boxed{\sum_{j\in\mathbf{J}} W_{ij} = h_i \qquad \forall i \in \mathbf{I}}$$

d) Now consider the constraint that states that we can have flow through a DC only if we locate at that candidate location. Formulate this constraint in terms of the notation above.

$$\boxed{V_j \le \left(\sum_{i\in\mathbf{I}} h_i\right) X_j \qquad \forall j \in \mathbf{J}}$$

Note that this formulation is LINEAR!!

e) Now consider the following different decision variables:

$$X_j = \begin{cases} 1 & \text{if we locate a DC at candidate site j} \\ 0 & \text{if not} \end{cases}$$

$$Y_{ij} = \begin{cases} 1 & \text{if we customer i is supplied by a DC at candidate site j} \\ 0 & \text{if not} \end{cases}$$

Formulate the objective function and constraints in terms of these new variables. In particular, the objective function will be the same. However, we will not explicitly need the constraint that the flow into a DC is equal to the flow out of the DC. We will need constraints that ensure that demand is satisfied and that we only use open DCs.

$$\begin{array}{lll} \text{Minimize} & \sum_{j \in \mathbf{J}} f_j X_j + \sum_{i \in \mathbf{I}} \sum_{j \in \mathbf{J}} \left(p + b_j + c_{ij}\right) h_i Y_{ij} & \\ \text{Subject to} & \sum_{j \in \mathbf{J}} Y_{ij} = 1 & \forall i \in \mathbf{I} \\ & Y_{ij} \le X_j & \forall i \in \mathbf{I}; \forall j \in \mathbf{J} \\ & X_j \in \{0,1\} & \forall j \in \mathbf{J} \\ & Y_{ij} \in \{0,1\} & \forall i \in \mathbf{I}; \forall j \in \mathbf{J} \end{array}$$

f) Will the two formulations give the same answer? Briefly justify your response.

Yes they will. In the first formulation the flow between the plant and any customer will always go through the open DC that has the smallest value of $\left(p + b_j + c_{ij}\right)$.

g) Which formulation do you think will be easier to solve? Again, briefly justify your answer.

The first formulation has fewer integer variables, but the Y_{ij} variables in the second need not be integer. (They will always turn out to be integer in any optimal solution, or they can be made integer without increasing the objective function value.) Also, the first formulation essentially has a big-*M*, constraint in part (d). Thus, on balance, the second formulation should be easier to solve, believe it or not.

Problem 3 (30 Percent): (Inventory)

You are in the office supplies business. You are concerned about ordering paper (assume for the moment that you stock only one form of paper – e.g., pads of 8 ½ by 11 lined paper). The fixed cost of placing an order with your supplier is \$50 when you consider the time and effort needed to place the order. The annual demand for pads of paper is 36,500 per year. Each pad of paper costs you \$0.80 and you have an inventory carrying cost rate of 20% per year.

a) Assuming that demand is deterministic and that you do not allow stockouts, find the optimal order size. Also, find the optimal time between orders (in months). *Hint: There is only one inventory model that we talked about that has deterministic demand.*

$$Q^* = \sqrt{\frac{2AK}{h}} = \sqrt{\frac{2(36500)50}{0.2 \bullet 0.8}} = 4{,}776.24$$

$$orders/year = \frac{36500}{4776} = 7.642$$

$$\text{time beteen orders} = \frac{12}{7.642} = 1.57 \text{ months}$$

b) What is the optimal cost? (inventory carrying cost and ordering cost)

$$\text{optimal cost} = \sqrt{2AKh} = \sqrt{2 \bullet 36500 \bullet 50 \bullet 0.16} = 764.20$$

c) Now suppose that you decide to order monthly instead of on the schedule suggested by your answer to part (a). What is the percentage increase in your inventory costs as a result of this decision?

Your order size will be $\hat{Q} = \frac{36500}{12} = 3041.67 = 0.6368Q^*$. So the ratio of the total costs is $\frac{1}{2}\left(\frac{1}{0.6368} + 0.6368\right) = 1.10355$, so the cost goes up by 10.355%

d) Now suppose that you also carry pens. Your projected annual sales of pens is 21,900 and each pen costs \$2.00 and has an inventory carrying cost of 20%. Again, the fixed cost of placing an order is \$50. Again, find the optimal order size and the optimal order interval in months.

$$Q^* = \sqrt{\frac{2AK}{h}} = \sqrt{\frac{2(21900)50}{0.2 \bullet 2}} = 2339.87$$

$$orders\,/\,year = \frac{21900}{2339.87} = 9.35949$$

$$\text{time beteen orders} = \frac{12}{9.35949} = 1.282 \text{ months}$$

e) What is the optimal cost associated with ordering pens? (inventory carrying cost and ordering cost)

$$\text{optimal cost} = \sqrt{2AKh} = \sqrt{2 \bullet 21900 \bullet 50 \bullet 0.4} = 935.95$$

f) Now suppose you can place a single order for both pens and pads of paper from the same supplier. The fixed cost of a joint order is \$60. Let *n* be the optimal number of orders per year (which we will allow to be fractional). Formulate the problem of finding the optimal number of orders per year to minimize the combined ordering cost, inventory carrying cost of pens and inventory carrying cost of pads of paper. *Note that this should be a non-linear optimization problem in terms of the single variable n.*

$$\begin{aligned}\text{Total cost} &= 60n + \frac{36500}{n} \bullet \frac{0.16}{2} + \frac{21900}{n} \bullet \frac{0.4}{2} \\ &= 60n + \frac{7300}{n}\end{aligned}$$

g) Solve the problem you formulated in part (f) for the optimal number of orders per year.

$$\frac{d\text{Total cost}}{dn} = 60 - \frac{7300}{n^2} = 0$$

$$n = \sqrt{\frac{7300}{60}} = 11.03$$

h) What is the total annual cost of joint ordering and how does it compare to the sum of the costs from parts (b) and (e)? Discuss briefly.

$$\text{Total cost} = 60n + \frac{7300}{n} = 60(11.03) + \frac{7300}{11.03} = 1323.63$$

Total cost with individual ordering is $764.20 + 935.95 = 1700.15$

or 28% more than the cost with joint ordering

Problem 4: (15 percent) (Linear programming):

You are in the business of producing plastic bottles. Such bottles can be used for any number of different uses. One application is as the plastic dispensers for windshield washer fluid in automobiles. You have four plants around the country that are capable of producing these containers. You sell these to the auto manufacturers. The unit shipment costs, production costs, capacities and demands are shown in the tables below. In all cases 1 unit is equal to 100 bottles.

Plant	Capacity (1000 bottles/ year)	Unit production cost ($/100 bottles)
Virginia	1750	40
Illinois	2250	35
Texas	2150	38
Oregon	2450	41
TOTAL	**8600**	

Auto Assembly Plant	Demand (100 bottles/ year)
Detroit	2500
St. Louis	1870
Los Angeles	2500
Baltimore	1630
TOTAL	**8500**

Shipment cost ($/100 bottles shipped)

	Detroit	St. Louis	Los Angeles	Baltimore
Virginia	50	80	240	10
Illinois	30	20	170	70
Texas	110	70	150	130
Oregon	190	160	40	240

Total Cost ($/100 bottles shipped)

	Detroit	St. Louis	Los Angeles	Baltimore
Virginia	90	120	280	50
Illinois	65	55	205	105
Texas	148	108	188	168
Oregon	231	201	81	281

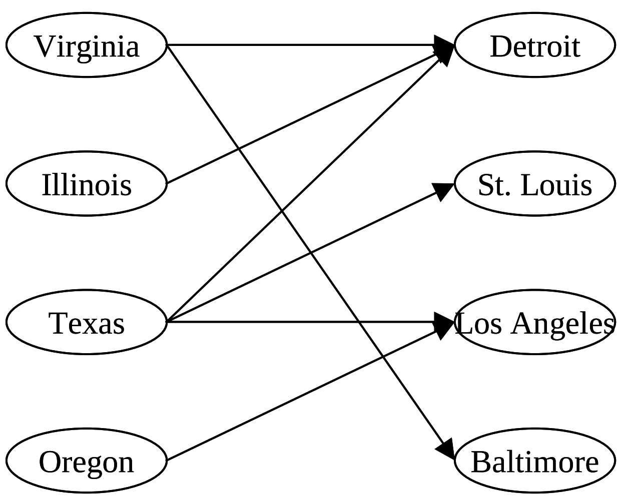
The problem of finding the minimum cost production and shipment plan can be solved as a linear programming problem. The solution is given below.

Flows (100s of bottles)

	Detroit	St. Louis	Los Angeles	Baltimore	Total Out	Capacity
Virginia	120	0	0	1630	**1750**	1750
Illinois	2250	0	0	0	**2250**	2250
Texas	130	1870	50	0	**2050**	2150
Oregon	0	0	2450	0	**2450**	2450
Total In	**2500**	**1870**	**2500**	**1630**		
Demand	2500	1870	2500	1630		

Total Cost $667,600

The figure below shows the non-zero flows.



a) What is the value of the dual variable associated with increasing the capacity in Texas?

0 since there is surplus supply at Texas

b) Suppose the demand in Los Angeles increased by 1 unit (i.e., 100 bottles). By how much would the objective function increase? What change in flows occurs? What is value of the dual variable associated with the demand in Los Angeles?

From	To	Flow change	Cost change
Texas	Los Angeles	+1	188
Total			188

Dual variable has a value of 188.

c) Now suppose the demand in Baltimore increases by 1 unit. By how much would the objective function increase? What change in flows occurs? What is the value of the dual variable associated with the demand in Baltimore?

From	**To**	**Flow change**	**Cost change**
Virginia	Baltimore	+1	50
Virginia	Detroit	-1	-90
Texas	Detroit	+1	+148
Total			108

Dual variable has a value of 108.