Please put your name on each page NOW. **Name** ____________________________

IE 310 Operations Research **Winter, 2003**
Mark S. Daskin **Dept. of IE/MS** **Northwestern University**

Introduction to Operations Research

QUIZ 2 – Solutions

Problem 1 (30 percent):

Recall that the location set covering model is to minimize the number of facilities that are needed subject to a constraint that every demand node is covered at least once. Toward this end, we can define the following inputs and set,

$$\mathbf{I} = \text{set of demand nodes}$$

$$\mathbf{J} = \text{set of candidate locations}$$

$$d_{ij} = \text{distance between nodes } i \text{ and } j$$

$$D_c = \text{coverage distance}$$

$$a_{ij} = \begin{cases} 1 & \text{if demand node } i \text{ can be covered by a facility at } j \\ 0 & \text{if not} \end{cases}$$

And the following decision variable:

$$X_j = \begin{cases} 1 & \text{if we locate at candidate site } j \\ 0 & \text{if not} \end{cases}$$

The location set covering model is then

$$\text{Minimize} \quad \sum_{j \in \mathbf{J}} X_j$$

$$\text{Subject to} \quad \sum_{j \in \mathbf{J}} a_{ij} X_j \geq 1 \qquad \forall i \in \mathbf{I}$$

$$X_j \in \{0,1\} \qquad \forall j \in \mathbf{J}$$

One of the important problems with the set covering model is that it does not discriminate well between alternative solutions. In the case of locating emergency medical service vehicles (ambulances), you might like a solution that (1) covers all demand at least once, and (2) from among the many solutions that cover all demand at least once you want to pick one that maximizes the number of nodes that are covered twice.

Let us define the following additional decision variable

$$S_i = \begin{cases} 1 & \text{if demand node i is covered at least two times} \\ 0 & \text{if not} \end{cases}$$

a) Using this notation, formulate a problem with **two** different objective functions. The first or **primary** objective is to minimize the number of facilities located. The second or **secondary** objective function is to maximize the number of demand nodes that are covered at least twice. The key constraint is that each demand node must be covered at least once.

$$\text{Minimize} \quad \sum_{j \in \mathbf{J}} X_j$$

$$\text{Maximize} \quad \sum_{i \in \mathbf{I}} S_i$$

$$\text{Subject to} \quad \sum_{j \in \mathbf{J}} a_{ij} X_j - S_i \geq 1 \qquad \forall i \in \mathbf{I}$$

$$X_j \in \{0,1\} \qquad \forall j \in \mathbf{J}$$

$$S_i \in \{0,1\} \qquad \forall i \in \mathbf{I}$$

b) Even this approach may not give us much discriminatory power since there may well be lots of solutions which, for example, use the minimum number of facilities and which also maximize the number of demand **nodes** that are covered twice. Suppose we now want our secondary objective to be that we maximize the number of **demands** that are covered at least twice. Let h_i be the demand at node *i*. Reformulate the secondary objective so that it now maximizes the number of demands that are covered twice.

$$\text{Minimize} \quad \sum_{j \in \mathbf{J}} X_j$$

$$\text{Maximize} \quad \sum_{i \in \mathbf{I}} h_i S_i$$

$$\text{Subject to} \quad \sum_{j \in \mathbf{J}} a_{ij} X_j - S_i \geq 1 \qquad \forall i \in \mathbf{I}$$

$$X_j \in \{0,1\} \qquad \forall j \in \mathbf{J}$$

$$S_i \in \{0,1\} \qquad \forall i \in \mathbf{I}$$

Please put your name on each page NOW. **Name** ________________________________

Problem 2 (35 percent):

The City of Yorksville is interested in building a new baseball stadium for its minor league team. It estimates that the incremental annualized cost of adding a bank of 25 seats to the stadium is \$500. The operating cost per game of the bank of (25) seats is \$25, whether or not the seats are used. The city plans to charge \$8 per person for each of the 15 games that the team will play in the stadium each year. For now, no other use of the stadium is anticipated. A consultant has estimated that the demand for the games **in units of 25 seats** is given by a Poisson distribution with a mean of 70. In other words, the mean number of seats sold per game is estimated to be $70 \bullet 25 = 1750$.

a) How large a stadium should the city build to maximize its expected profit from the stadium? Be careful of the units here. Note that the stadium can only be built in increments of 25 seat blocks. You should probably work in these units and then just multiply your answer (which will be the number of 25-seat blocks to build) by 25 at the very end.
Hint: write down the expected marginal profit associated with increasing the number of blocks of 25 seats from q to q+1. Also, at the end of the entire exam is a table giving the cumulative Poisson distribution for λ*=70.*

$$\begin{aligned} & \text{Increase from q to q} + 1 \text{ banks of } 25 \text{ seats as long as} \\ EMR(q \to q+1) &= -500 + P(D > q) \bullet [8 * 25 - 25] \bullet 15 + P(D \le q) \bullet [-25] \bullet 15 \\ &= -500 - 375 + [1 - P(D \le q)]3000 \\ &= 2125 - 3000 P(D \le q) \ge 0 \\ & \text{or} \\ P(D \le q) &\ge \frac{2125}{3000} = 0.708\overline{3} \end{aligned}$$

So we build 74 banks of 25 seats of a stadium with a capacity of 1850.

b) Now suppose that the city wants to account for the extra revenue that it will make on the sales tax due to concessions (food, beverages, souvenirs, parking). It estimates that the sales tax will bring in an additional \$1.50 per person attending the game. How does this affect the decision about how many seats to build? What is the new optimal capacity of the stadium?

$$
\begin{aligned}
&\text{Increase from q to q}+1\text{ banks of }25\text{ seats as long as}\\
EMR(q \to q+1) &= -500 + P(D > q) \bullet [(8+1.5)*25 - 25] \bullet 15 + P(D \le q) \bullet [-25] \bullet 15\\
&= -500 - 375 + [1 - P(D \le q)]3562.5\\
&= 2687.5 - 3562.5P(D \le q) \ge 0\\
&\text{or}
\end{aligned}
$$

$$
P(D \le q) \ge \frac{2687.5}{3562.5} = 0.7544
$$

So we build a stadium with a capacity of 76*25=1900 seats.

<u>Problem 3 (35 percent):</u>

WBBM, a local radio station, runs a "Smart Quiz" four times each morning during the five weekday mornings. The seventh caller is given a chance to answer the question of the hour. If the answer is correct, the caller receives a prize which is donated by one of WBBM's advertisers. For example, a jeweler might sponsor the Smart Quiz for a week (20 questions) and might offer a $200 watch for every caller who gets the correct answer. An airline might offer a pair of free round-trip tickets to any destination to which the airline flies. Note that there can be at most one winner for each question since they only take the seventh caller.

a) Suppose the probability of someone getting the answer right is 0.4. What is the distribution of the number of correct answers in a week?

Binomial with n=20 and p=0.4

b) What is the average number of correct answers in a week?

8

c) What is the variance of the number of correct answers in a week?

4.8

d) Suppose a sponsor (advertiser) is considering sponsoring the Smart Quiz for four weeks. What is the distribution of the number of correct answers, the mean of the number of correct answers and the variance of the number of correct answers in a four-week period?

Binomial with n=80, p=0.4, Expected number =32 and variance of the number of correct answers equal to 19.2

e) If the prize that the sponsor is giving is worth \$200 for each winner (e.g., a \$200 watch), what is the approximate probability that the total value of all prizes awarded by the sponsor in a four-week period will exceed \$8000. Justify your answer.
Hint: You might find the cumulative standard Normal distribution given at the end of the exam useful here.
Be sure to show your work so I know how you got the answer you obtained.

$$\begin{aligned} P(value > 8000) &= P\left(\frac{value - 32 \bullet 200}{200\sqrt{19.2}} > \frac{8000 - 32 \bullet 200}{200\sqrt{19.2}}\right) \\ &= P(Z > 1.826) \\ &\approx 0.034 \end{aligned}$$

or it is the probability that there will be more than 40 winners which is 0.027 using the binomial table in Excel (which I do not expect you to do). Or we can approximate the answer as follows

$$\begin{aligned} P(value > 8000) &= P(\#\text{ winners} > 40) \\ &= P\left(\frac{\#\text{ winners} - 32}{\sqrt{19.2}} > \frac{40.5 - 32}{\sqrt{19.2}}\right) && \text{using the continuity correction} \\ &= P(Z > 1.94) \\ &\approx 0.026 && \text{using interpolation} \end{aligned}$$

Any reasonable answer in this range is fine as long as you show your work.

Please put your name on each page NOW. **Name** ____________________

Cumulative Poisson for λ=70

j	P(D≤j)	j	P(D≤j)
50	0.0075	75	0.74817
51	0.0107	76	0.78377
52	0.0151	77	0.81613
53	0.0208	78	0.84517
54	0.0282	79	0.87090
55	0.0377	80	0.89342
56	0.0495	81	0.91288
57	0.0641	82	0.92949
58	0.0816	83	0.94350
59	0.1024	84	0.95517
60	0.1267	85	0.96479
61	0.1545	86	0.97261
62	0.1860	87	0.97891
63	0.2209	88	0.98392
64	0.2591	89	0.98786
65	0.3003	90	0.99092
66	0.3439	91	0.99328
67	0.3895	92	0.99507
68	0.4365	93	0.99642
69	0.4841	94	0.99742
70	0.5317	95	0.99816
71	0.5787	96	0.99870
72	0.6243	97	0.99909
73	0.6681	98	0.99937
74	0.7095	99	0.99957
75	0.7482	100	0.99971

Please put your name on each page NOW. **Name** ________________________

Cumulative standard Normal Distribution Z~N(0,1)

Zα	P(Z≤Za)	Zα	P(Z≤Za)
0.00	0.5000	1.25	0.8944
0.05	0.5199	1.30	0.9032
0.10	0.5398	1.35	0.9115
0.15	0.5596	1.40	0.9192
0.20	0.5793	1.45	0.9265
0.25	0.5987	1.50	0.9332
0.30	0.6179	1.55	0.9394
0.35	0.6368	1.60	0.9452
0.40	0.6554	1.65	0.9505
0.45	0.6736	1.70	0.9554
0.50	0.6915	1.75	0.9599
0.55	0.7088	1.80	0.9641
0.60	0.7257	1.85	0.9678
0.65	0.7422	1.90	0.9713
0.70	0.7580	1.95	0.9744
0.75	0.7734	2.00	0.9772
0.80	0.7881	2.05	0.9798
0.85	0.8023	2.10	0.9821
0.90	0.8159	2.15	0.9842
0.95	0.8289	2.20	0.9861
1.00	0.8413	2.25	0.9878
1.05	0.8531	2.30	0.9893
1.10	0.8643	2.35	0.9906
1.15	0.8749	2.40	0.9918
1.20	0.8849	2.45	0.9929
1.25	0.8944	2.50	0.9938