Please place your name on each page Name ______________________________

IE 310 – Introduction to Operations Research

QUIZ 2 – SOLUTIONS

Problem 1 (40 Percent):

You are beginning a new executive jet service in which senior executives can rent one of your planes along with a pilot for $15,000 per day. It costs you $10,000 per day to lease the plane and to hire the pilot for the plane. The catch is that you need to lease the plane on a long-term contract and hire pilots on annual contracts, before you know how many aircraft rentals you will have each day.

You estimate that the distribution of the number of requested rentals per day is Poisson with a rate of 8 planes per day.

a) Complete the table below.

Number	Probability	Cumulative Probability
0	0.00034	0.00034
1	0.00268	0.00302
2	0.01073	0.01375
3	0.02863	0.04238
4	0.05725	0.09963
5	0.09160	0.19124
6	0.12214	0.31337
7	0.13959	0.45296
8	0.13959	0.59255
9	0.12408	0.71662
10	0.09926	0.81589
11	0.07219	0.88808
12	0.04813	0.93620

b) Compute the cost of overage (the cost of having too many planes on any day) and the cost of underage (the cost of having too few planes on any given day.

The overage cost is $10,000 and the underage cost is $5,000.

c) Find the optimal number of planes to lease.

The optimal number of planes is given by the largest value of Q such that the probability that the demand is less than or equal to Q is greater than or equal to 1/3. This means that the optimal number of planes is 7 planes. ***(See Denardo, p. 404).***

d) You can compute the expected profit using the table below. Note that the expected number of planes rented for any given number owned can be computed using the following formula

$$E(\text{number rented}) = \sum_{j=0}^{\#\text{ owned -1}} jP(\text{demand} = j) + (\#\text{ owned})P(\text{demand} \geq \#\text{ owned})$$

Thus, if you own 4 planes, the expected number rented is given by

$$0(0.00034) + 1(0.00268) + 2(0.01073) + 3(0.02863) + 4(1 - 0.04238) = 3.94051$$

Complete the table below to (1) verify your answer to part (c) and

(2) to find the maximum expected profit.

Demand	Probability	Cumulative Probability	1-Cumulative	Number owned	Expected number rented	Cost	Expected revenue	Expected Profit
0	0.00034	0.00034	0.99966	0	0.00000	$ -	$ -	$ -
1	0.00268	0.00302	0.99698	1	0.99966	$ 10,000.00	$ 14,994.97	$ 4,994.97
2	0.01073	0.01375	0.98625	2	1.99665	$ 20,000.00	$ 29,949.68	$ 9,949.68
3	0.02863	0.04238	0.95762	3	2.98289	$ 30,000.00	$ 44,743.37	$ 14,743.37
4	0.05725	0.09963	0.90037	4	3.94051	$ 40,000.00	$ 59,107.67	$ 19,107.67
5	0.09160	0.19124	0.80876	5	4.84088	$ 50,000.00	$ 72,613.18	$ 22,613.18
6	0.12214	0.31337	0.68663	6	**5.64964**	**$ 60,000.00**	**$ 84,744.64**	**$ 24,744.64**
7	0.13959	0.45296	0.54704	7	**6.33627**	**$ 70,000.00**	**$ 95,044.03**	**$ 25,044.03**
8	0.13959	0.59255	0.40745	8	**6.88331**	**$ 80,000.00**	**$ 103,249.62**	**$ 23,249.62**

e) Your firm is in a growth mode and you realize that there is an implicit penalty for not being able to satisfy demand when customers want aircraft as each such turndown creates ill will. How large must this implicit penalty be (in dollars per day) for you to justify leasing one additional aircraft?

The implicit penalty would essentially add to the cost of underage. Let *P* be the penalty. Then we want:

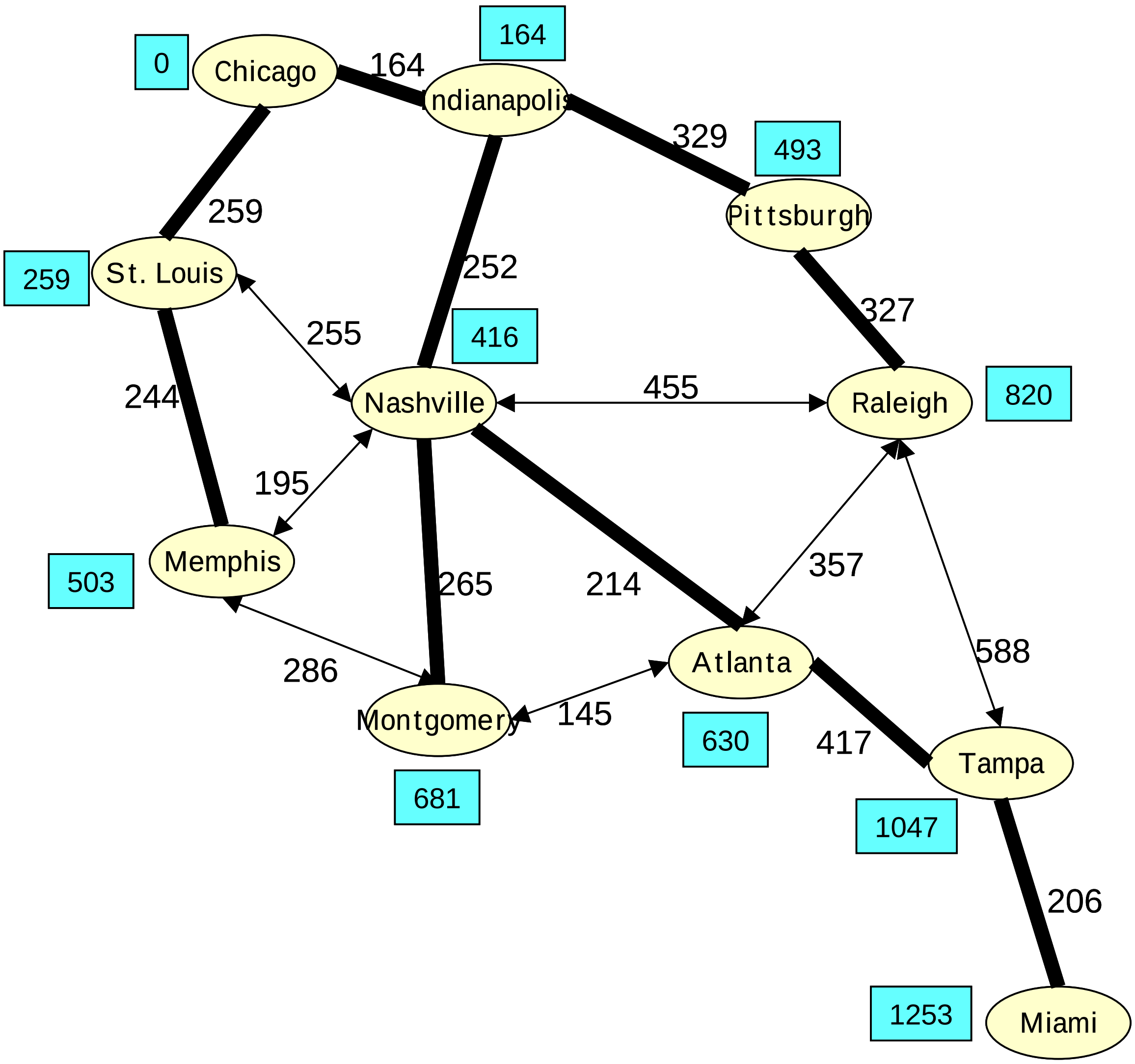
$$\begin{aligned}\frac{5000+P}{15000+P} &> 0.45296\\ 5000+P &> 0.45296(15000+P)\\ 5000+P &> 6794.4+0.45296P\\ 0.54704P &> 1794.4\\ P &> 3280.20\end{aligned}$$

Problem 2 (20 Percent):

You want to spend Spring break in Miami. (Not at all clear that is a good idea!) For the following network, find the minimum path tree rooted at Chicago.

Note that the figure is NOT drawn to scale. Clearly show the shortest path and the distance from Chicago to each node on the path.

Please place your name on each page **Name** ______________________________



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Problem 3 (40 Percent):

Consider the following problem of finding the approximating the optimal number of facilities needed to serve a given area. For example, we might want to find the number of computer service centers needed to serve the continental United States. The repair personnel will travel from a facility to a customer site. Our (initial) objective is to minimize the total fixed cost of the facilities and the travel cost of the repair personnel.

We can show that (under appropriate assumptions which are beyond the scope of this quiz) the total cost as a function of the number of facilities, *N*, is given by

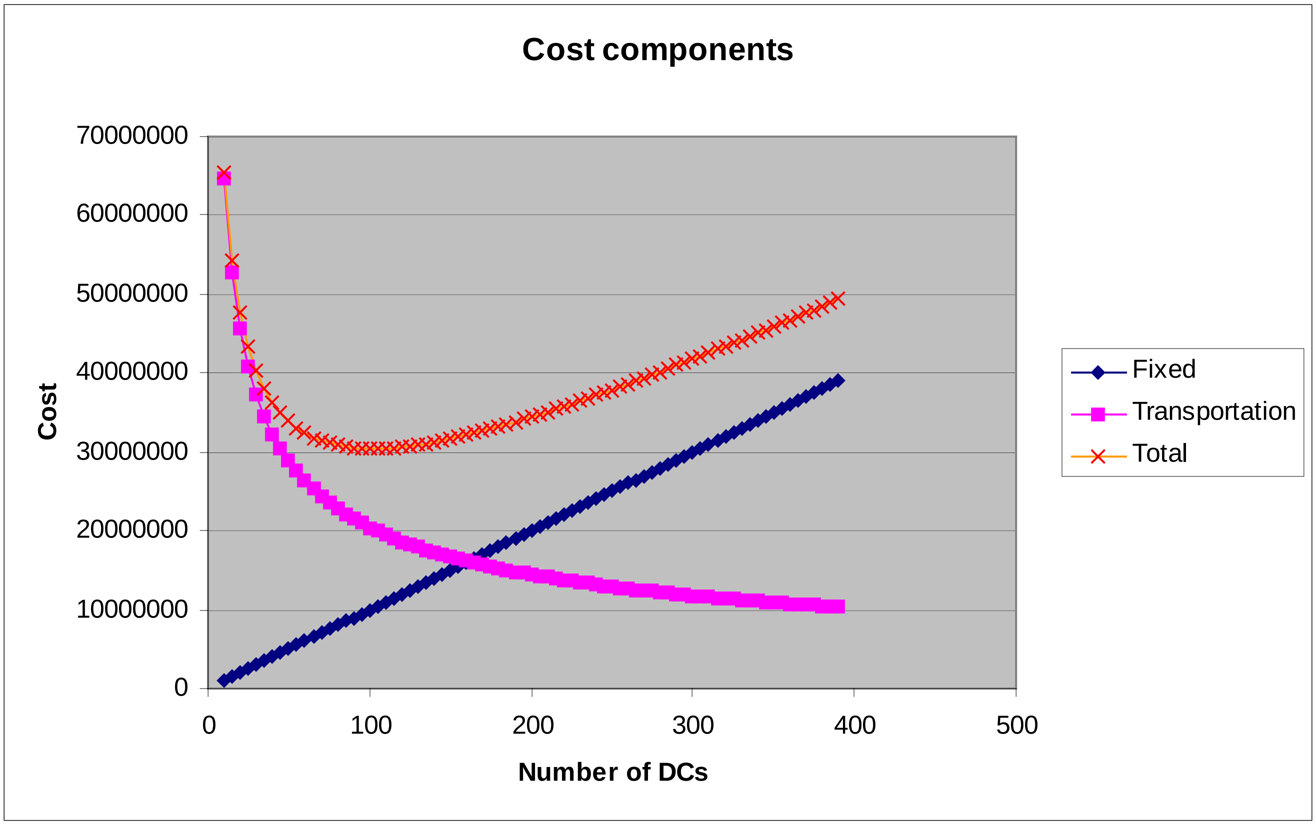
$$TC(N) = fN + \frac{\beta D 2\sqrt{A}}{3\sqrt{2}} \frac{1}{\sqrt{N}}$$

where

f	fixed annual cost of locating a facility
β	cost per mile of traveling between a facility and a customer site
D	annual demand for services
A	area (in square miles) of the region being served

The first term fN represents the total fixed cost of the facilities and the second term $\frac{\beta D 2\sqrt{A}}{3\sqrt{2}} \frac{1}{\sqrt{N}}$ represents the total travel cost of the repair personnel to the customers.

a) On the graph below, plot (i) the fixed cost as a function of the number of facilities located, (ii) the total travel cost to the customers and (iii) the total cost



b) Derive an equation for the optimal number of service facilities as a function of the key inputs: the fixed cost of the facilities, the area to be served, the unit transportation cost, and the annual demand.

$$TC(N) = fN + \frac{\beta D 2\sqrt{A}}{3\sqrt{2}} \frac{1}{\sqrt{N}}$$

$$\frac{dTC(N)}{dN} = f - \frac{1}{2}\frac{\beta D 2\sqrt{A}}{3\sqrt{2}} \frac{1}{N^{3/2}} = f - \frac{\beta D\sqrt{A}}{3\sqrt{2}} \frac{1}{N^{3/2}} = 0$$

$$f 3\sqrt{2}\, N^{3/2} = \beta D\sqrt{A}$$

$$N^{3/2} = \frac{\beta D\sqrt{A}}{f 3\sqrt{2}}$$

$$N = \left(\frac{\beta D\sqrt{A}}{f 3\sqrt{2}}\right)^{2/3} = \left(\frac{\beta D}{3f}\right)^{2/3} \left(\frac{A}{2}\right)^{1/3}$$

c) For the following inputs, find (i) the optimal number of service centers and (ii) the total cost when using that number of service centers.

$f = 100{,}000$

$A = 3{,}000{,}000$ approximately the area of the United States

$D = 500{,}000$

$\beta = 0.50$

$$N = \left(\frac{\beta D\sqrt{A}}{f3\sqrt{2}}\right)^{2/3} = \left(\frac{\beta D}{3f}\right)^{2/3}\left(\frac{A}{2}\right)^{1/3}$$

$$N = \left(\frac{0.5\bullet 500000}{3\bullet 100000}\right)^{2/3}\left(\frac{3000000}{2}\right)^{1/3}$$

$N = 101.370003$

$\text{N} = 101 \text{ gives TC} = 30{,}411{,}212$

$\text{N} = 102 \text{ gives TC} = 30{,}411{,}403$

So use $N = 101$

d) Note that this model is similar to the EOQ model except that the transport cost varies as $1/\sqrt{N}$, while the annual order cost varies as $1/Q$ in the EOQ model. In other words, here we have a square root term in the denominator while the EOQ model does not have the square root in the denominator.

In the EOQ model, we know that at the optimal solution, the fixed costs equal the holding costs. **For this model, what is the ratio of the fixed facility costs to the transport costs at the optimal solution?**

$$N^* = \left(\frac{\beta D}{3f}\right)^{2/3}\left(\frac{A}{2}\right)^{1/3}$$

$$TC\left(N^*\right) = f\,N^* + \frac{\beta D 2\sqrt{A}}{3\sqrt{2}}\frac{1}{\sqrt{N^*}}$$

$$f\left(\frac{\beta D}{3f}\right)^{2/3}\left(\frac{A}{2}\right)^{1/3} + \frac{\beta D\sqrt{2A}}{3f}\left(\frac{3f}{\beta D}\right)^{1/3}\left(\frac{2}{A}\right)^{1/6}$$

$$= \left(\frac{\beta D}{3}\right)^{2/3}\left(\frac{Af}{2}\right)^{1/3} + 2\left(\frac{\beta D}{3}\right)^{2/3}\left(\frac{Af}{2}\right)^{1/3}$$

$$= 3\left(\frac{\beta D}{3}\right)^{2/3}\left(\frac{Af}{2}\right)^{1/3}$$

So the ratio of the fixed facility costs to the transport costs is 1:2.

e) Your firm now realizes that you also need to maintain an inventory of repair parts at each of the service facilities. This will add another term to the total cost. The inventory at **each** facility is proportional to the square root of the demand assigned to or served by the facility. This demand is approximately D/N. Thus, the inventory cost at each facility is given by $\rho\sqrt{D/N}$ for some suitably defined value of ρ. The **total** inventory cost is then given by $\rho\sqrt{DN}$.

Will the inclusion of inventory costs **increase**, **decrease**, or **leave unchanged** the optimal number of facilities?

Note that you should be able to answer this without doing any math and just using your intuition.

This will tend to <u>decrease</u> the number of facilities since it introduces a term that exhibits economies of scale in *N*.