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IE 310 Operations Research
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Introduction to Operations Research

QUIZ 1 – SOLUTIONS

Problem 1 (40 percent):

You just bought 5000 square feet of retail space and plan to sell high-end electronic equipment. Based on industry reports you **initially** estimate the monthly contribution per square foot for each of four types of electronic equipment as shown below.

In addition, you want there to be some diversity in the equipment that you sell and so you have decided to set a maximum allocation of floor space to each of the different types of equipment. This is also shown in the table.

Equipment	Contribution per square foot	Maximum desired space
VCRs	15	2500
Cameras	25	800
TVs	12	3500
Sound systems	11	3000

Define the following decision variables:

X_{VCR} number of square feet allocated to VCR sales
X_{Camera} number of square feet allocated to camera sales
X_{TV} number of square feet allocated to TV sales
X_{Sound} number of square feet allocated to sound system sales

a) Write the objective function of maximizing the total contribution in terms of the four decision variables

Maximize $15\,X_{VCR} + 25\,X_{Camera} + 12\,X_{TV} + 11\,X_{Sound}$

b) Write the four constraints on the maximum allowable space for each of the four types of items.

$$
\begin{aligned}
X_{VCR} &\le 2500\\
X_{Camera} &\le 800\\
X_{TV} &\le 3500\\
X_{Sound} &\le 3000
\end{aligned}
$$

c) Write the constraint on the total floor space available in terms of the four decision variables

$$X_{VCR} + X_{Camera} + X_{TV} + X_{Sound} \le 5000$$

d) Write the non-negativity constraints

$$
\begin{array}{llllr}
X_{VCR} & & & & \geq 0 \\
& X_{Camera} & & & \geq 0 \\
& & X_{TV} & & \geq 0 \\
& & & X_{Sound} & \geq 0
\end{array}
$$

e) The **optimal solution** to this problem is given in the following table. Briefly describe the optimal solution in words as a **policy** that would tell you what to do if you had somewhat more space or somewhat less space. *I do not want a description that says the obvious, "Allocate 2500 square feet to VCRs, 800 square feet to cameras, and 1700 square feet to TVs."*

Equipment	Square feet allocated
VCRs	2500
Cameras	800
TVs	1700
Sound systems	0

Use as many square feet as possible for VCRs and Cameras and then use any remaining area for TVs. Do not allocate any space to sound systems.

f) The **optimal objective** function value is 77900. Using your knowledge of linear programming and the optimal solution shown in part (e)

- Complete the following table of dual variables based on your knowledge of the relationship between the primal and the dual problems.
- Briefly justify each of your three answers.

Constraint	Dual variable
VCR Space	3
Camera Space	13
TV Space	0
Sound Space	0
Total Space	12

The dual variable associated with TV and sound space must be 0 since we do not use all the available TV and sound space. The dual variable associated with the total space constraint must be such that $77900 = 3*2500 + 13*800 + DUAL_{total}*5000$,

so $DUAL_{total} = \dfrac{77900 - 3*2500 - 13*800}{5000} = 12$

g) After thinking about the solution, you realize that the allocation of space to VCRs and to TVs is inappropriate since sales of these two types of equipment are related. You decide that you do not want the space allocated to VCRs to be more than half the space allocated to TVs. Write this constraint in terms of the decision variables defined above.

$X_{VCR} \leq 0.5 X_{TV}$ or $X_{VCR} - 0.5 X_{TV} \leq 0$ or $2X_{VCR} - X_{TV} \leq 0$

h) Will the addition of this constraint result in an increase, decrease, or no change in the optimal objective function value? Briefly justify your answer.

Addition of this constraint will make the optimal objective function value worse (or smaller since this is a maximization problem).

Problem 2 (30 percent):

Suppose you are in the business of producing boxes of corn flakes. You have two plants: one in Boston and a newer, slightly more efficient one in Denver. The cost per box (in cents) of producing during regular hours is given below for the two plants. Also shown in the table is the maximum allowable daily production during regular shifts

	Cost per box produced	Maximum regular shift production
Boston	120	6500
Denver	100	7000

In addition, you estimate the following shipment costs per box of cereal between these two plants and the four major markets that you want to supply. Again, these costs are in cents per box.

	New York	Chicago	Dallas	Los Angeles
Boston	2	9	15	26
Denver	16	9	7	8
Daily Demand	4000	4000	3500	6000

The table also shows the daily demand in each of the four markets

Finally, you can produce additional amounts at each of the two plants at an increase in unit cost of 10%. The maximum additional amount that you can produce at each of the two plants is 50% of the amount you can produce during the regular shifts (e.g., 3250 at Boston).

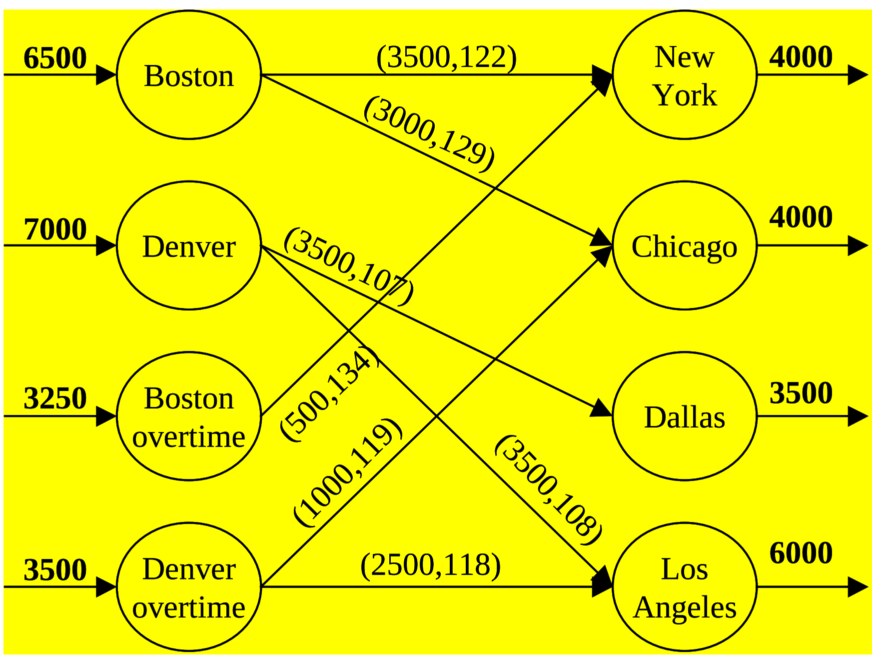
The optimal solution to the problem is shown in the table below:

		Destinations			
		New York	Chicago	Dallas	Los Angeles
Origins	Boston	3500	3000	0	0
	Denver	0	0	3500	3500
	Boston overtime	500	0	0	0
	Denver overtime	0	1000	0	2500

Actual out		Maximum out
6500	=<=	6500
7000	=<=	7000
500	<=	3250
3500	=<=	3500

	New York	Chicago	Dallas	Los Angeles
Actual in	4000	4000	3500	6000
	=>=	=>=	=>=	=>=
Required in	4000	4000	3500	6000

a) Complete the following diagram showing (i) the maximum allowable flows into each "production" node, (ii) the required flows out of each "demand" node, (iii) the **actual** flows between each supply and demand node and (iv) the costs between production cities and demand cities. Do **not** show flows that are zero. *Note that the values for the flows (3500) and unit cost (122) between Boston regular time production and New York are shown in the way they should be displayed.*



The dual variables for the problem are shown in the table below.

	Dual Variable
Boston	-12
Denver	-32
Boston overtime	0
Denver overtime	-22

New York	134
Chicago	141
Dallas	139
Los Angeles	140

b) Suppose that you can produce an extra 10 units during overtime in Denver (above and beyond the 3500 you currently can produce). What is the change in the objective function value going to be?

The objective function value will go down by 220 or 22 times 10.

c) How will the flows change if the maximum overtime production in Denver increases by 10 units?

From	To	Flow change	Unit cost	Total cost change
Denver overtime	Chicago	10	119	1190
Boston	Chicago	-10	129	-1290
Boston	New York	10	122	1220
Boston overtime	New York	-10	134	-1340
Total				-220

d) Now suppose that you can sell the marginal additional (last few) boxes of cereal in Chicago for the amounts shown in the table below.

This table reflects the saturation of the Chicago market. The distributor is willing to buy 10 more boxes at $1.50 each, after that he is only willing to pay $1.46 for boxes 11 through 19, $1.42 for boxes 20 through 27 and so on since she feels she cannot sell much more in the Chicago market.

Price	Additional sales
$1.50	10
$1.46	9
$1.42	8
$1.38	7
$1.34	6
$1.30	5

Using your knowledge of linear programming,

- how many **more** boxes should you plan to sell in Chicago assuming you want to maximize the profit that you obtain from the Chicago market?
- How much profit (sales price minus cost) will these sales generate?

(You can assume that the basic structure of the solution will not change if the flow from the Boston Overtime node to the New York demand node increases all the way from 500 to 3250.)

The dual variable for Chicago tells you that each additional unit that you ship to Chicago costs you $141. Note that this is NOT the direct shipment cost to Chicago from either of the sources supplying Chicago (Boston or Denver Overtime). However, we would get this 141 by the following changes:

From	To	Flow change	Unit cost	Total cost change
Boston	Chicago	1	129	129
Boston	New York	-1	122	-122
Boston overtime	New York	1	134	134
Total				141

Thus, if the revenue exceeds $141 then it is desirable to sell the boxes of cereal; if not, you lose money on every new sale. Thus, you would sell 10 boxes at $1.50, 9 at $1.46, and 8 at $1.42 for a total of $39.50 for the 27 boxes. Producing and shipping these 27 boxes costs $1.41*27 or $38.07. Our net profit is therefore $39.50-$38.07 or $1.43.

<u>Problem 3: (30 percent):</u>

You are interested in scheduling nurses on both the general surgery floor and the intensive care unit (ICU). There are two types of nurses you can schedule: general surgery nurses and ICU-trained nurses. ICU-trained nurses can work on both the ICU and on the general surgery floor, but general surgery nurses can not work in the ICU because of their lower level of training. General surgery nurses are paid $160 for an 8-hour shift and ICU-trained nurses are paid $220 per shift whether they work on the ICU or the general surgery floor.

A nurse who starts an 8-hour shift at the beginning of time hour t, works during hours t, $t+1$, $t+2$, $t+3$, $t+5$, $t+6$, and $t+7$. The nurse does **not** work during hour $t+4$ which is his/her meal break. Define the following input:

a_{ij} 1 if a nurse who begins work at the beginning of hour i is available to work during hour j; 0 if not.

Let us also define the following additional inputs

g_j Number of nurses needed in general surgery during hour *j*

h_j Number of nurses needed in the ICU during hour *j*

and the following decision variables:

X_i number of general surgery nurses who begin a shift at the beginning of hour *i*

Y_i number of ICU-trained nurses who begin a shift at the beginning of hour *i*

Z_j number of ICU-trained nurses who are **working in the ICU** during hour *j*. You can assume that any ICU nurses who are available for work during hour *j* but who are not assigned to the ICU are assigned to the general surgery floor during hour *j*.

What you want to do is to determine a schedule that minimizes the total cost of staffing both the ICU and the general surgery floor. Clearly this should be the total number of general surgery nurses who start during a day times \$160 plus the total number of ICU nurses who start during a day times \$220.

Assume throughout the problem that there are 24 hours during the day and that a nurse who begins late in the evening (e.g., at 11 p.m.) can satisfy the need for nurses in the early morning (e.g., at midnight and 1 a.m. and so on). In other words, assume that time wraps around the day and that you are scheduling "typical" day.

a) Using the notation above, write the objective function of minimizing the total cost.

$$\textbf{Minimize } 160\sum_{i=1}^{24} X_i + 220\sum_{i=1}^{24} Y_i$$

b) Write down the constraint that says that relates the number of ICU nurses assigned to work in the ICU in hour *j* to the number of ICU-trained nurses who start in various hours. This should relate the Z_j variables to the Y_i variables.

$$Z_j \le \sum_{i=1}^{24} a_{ij} Y_i \qquad j = 1,\ldots,24$$

c) Write the constraint that states that the number of ICU-trained nurses assigned to the ICU in hour *j* must meet the minimum requirement for hour *j*.

$$Z_j \ge h_j \qquad\qquad j = 1,\ldots,24$$

d) Write a constraint that states that the number of nurses assigned to the general surgery floor in hour *j* must meet the minimum requirement for hour *j*. Note that this should relate the X_i variables (and probably the Y_i and Z_j variables) to the g_j inputs.

$$\underbrace{\sum_{i=1}^{24} a_{ij} X_i}_{\text{number of general nurses available in hour j}} + \underbrace{\left(\sum_{i=1}^{24} a_{ij} Y_i - Z_j\right)}_{\text{number of ICU-trained nurses not assigned to the ICU in hour j and therefore available for assignment to general surgery floors}} \geq g_j \qquad j = 1,\ldots,24$$

e) Write down the non-negativity constraints.

$$X_i \geq 0 \qquad i = 1,\ldots,24$$
$$Y_i \geq 0 \qquad i = 1,\ldots,24$$
$$Z_j \geq 0 \qquad j = 1,\ldots,24$$