Please place your name on each page **Name** **SOLUTIONS**

IE 310 – Introduction to Operations Research

FINAL EXAMINATION – SOLUTIONS

Problem 1 (25 Percent):

You are in the business of producing high-end laptop computers. You have five different models that you can sell with the following characteristics:

Characteristic\Model	Employee	Manager	VP	President	CEO
Monitor	15”	15”	17”	17”	19”
Drive	40 GB	60 GB	60 GB	120GB	120GB
Memory	2 GB	3 GB	3GB	3 GB	5 GB
Production cost	$1000	$1200	$1350	$1500	$1700
Sale Price	$1200	$1500	$1700	$2000	$2400
Margin or Contribution per machine	$200	$300	$350	$500	$700

Clearly, the three monitors are different from each other as are the disk drives. However, the 5 GB of memory in the CEO model is actually composed of two chips: a 3 GB chip and a 2 GB chip. The maximum weekly supply of each of the different components from your suppliers is given below:

Component	Style	Weekly supply
Monitor	15”	200
	17”	150
	19”	75
Drive	40 GB	200
	60 GB	300
	120 GB	200
Memory	2 GB	300
	3 GB	325

Define the following decision variables:

X_E, X_M, X_V, X_P, X_C The number of Employee, Manager, VP, President and CEO models built each week.

Assume that sales of each model are limited only by the supply capacity; in other words, assume that you can sell all of any model that you can produce and still get the indicated margin or contribution for each machine.

a) Write down the objective function of maximizing the total contribution in terms of these decision variables

$$\textit{Maximize} \qquad 200\,X_E + 300\,X_M + 350\,X_v + 500\,X_P + 700\,X_C$$

b) Write the constraints that relate the production quantities to the number of available monitors of each size.

$$\begin{array}{llllllll}
X_E & +X_M & & & & \le & 200 \\
 & & X_V & +X_P & & \le & 150 \\
 & & & & X_C & \le & 75
\end{array}$$

c) Write the constraints that relate the production quantities to the number of available disk drives of each size.

$$\begin{array}{llllllll}
X_E & & & & & \le & 200 \\
 & X_M & +X_V & & & \le & 300 \\
 & & & X_P & +X_C & \le & 200
\end{array}$$

d) Write the constraints that relate the production quantities to the number of available memory chips of each size.

$$\begin{array}{llllllll}
X_E & & & & +X_C & \le & 300 \\
 & X_M & X_V & +X_P & +X_C & \le & 325
\end{array}$$

e) The optimal solution to the problem is to produce the following quantities each week

Model	**Optimal production quantity**
Employee	100
Manager	100
VP	25
President	125
CEO	75

In addition, the dual variables associated with the various constraints are given below

Component	Type	Dual variable
Monitors	15”	-200
	17”	-250
	19”	-450
Drives	40 GB	0
	60 GB	0
	120 GB	-150
Memory	2 GB	0
	3 GB	-100

Using this information and the constraints above, how would the production plan change if you had one more 17” monitor? **Justify** your answer. Be sure that your answer agrees with the dual variable above.

You would **increase** the number of VP laptops by 1 (increase in o.f. of 350)
Decrease the number of Manager laptops by 1 (decrease in o.f. of 300)
Increase the number of Employee laptops by 1 (increase in o.f. of 200)

For a net increase in the objective function of 250 in accord with the dual variable.

f) How would the production plan change if you had one more 60 GB hard drive? **Justify** your answer.

No change in production since you are not using all the 60 GB hard drives that you have available.

g) Suppose we add a constraint that we have to produce at least 150 President laptops. Will the total contribution increases or decrease? **Justify** your answer.

The total contribution will **decrease** since we are adding a constraint and the new constraint will clearly be binding.

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<u>Problem 2:</u>

You are interested in locating fire stations in Chicago. Your objective is to minimize the demand-weighted average (or total since the number of demands per unit time will be treated as a known constant) response time. However, you also want to be sure that no area experiences a response time that exceeds 5 minutes.

Define the following notation:

Inputs and sets

I	set of demand nodes indexed by *i*
J	set of candidate fire station locations indexed by *j*
P	number of fire stations to locate
t_{ij}	response time between candidate site *j* and demand node *i*
h_i	average number of demands per unit time in demand area *i*
T^c	maximum allowable (critical) time (5 minutes in this case, but use this notation to keep the problem statement more general)

Decision variables

$$X_j \quad \begin{cases} 1 & \text{if we locate at candidate site } j \\ 0 & \text{if not} \end{cases}$$

$$Y_{ij} \quad \begin{cases} 1 & \text{if demand node } i \text{ is assigned to candidate site } j \\ 0 & \text{if not} \end{cases}$$

Throughout the formulation sub-parts of the question below be sure to indicate the sets over which you are summing and/or the variables and sets to which constraints apply.

a) Using this notation, formulate the objective function of minimizing the demand weighted total response time

$$Min \quad \sum_{j \in J} \sum_{i \in I} h_i t_{ij} Y_{ij}$$

b) Formulate the constraint that each demand node is assigned to exactly one facility (fire station)

$$\sum_{j \in J} Y_{ij} = 1 \qquad \forall i \in I$$

c) Formulate the constraint that each demand node can only be assigned to an open candidate site (fire station)

$$Y_{ij} \le X_j \qquad \forall i \in I; \forall j \in J$$

d) Formulate the constraint that you locate exactly *P* fire stations

$$\sum_{j \in J} X_j = P$$

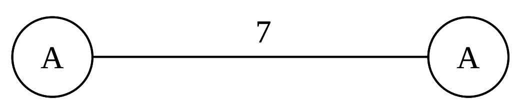
e) Formulate the constraint that each demand node must be assigned to a fire station that is no more than T^c minutes away.

$$\sum_{j \in J} t_{ij} Y_{ij} \le T^c \qquad \forall i \in I$$

f) Formulate the integrality constraints on the location and assignment variables

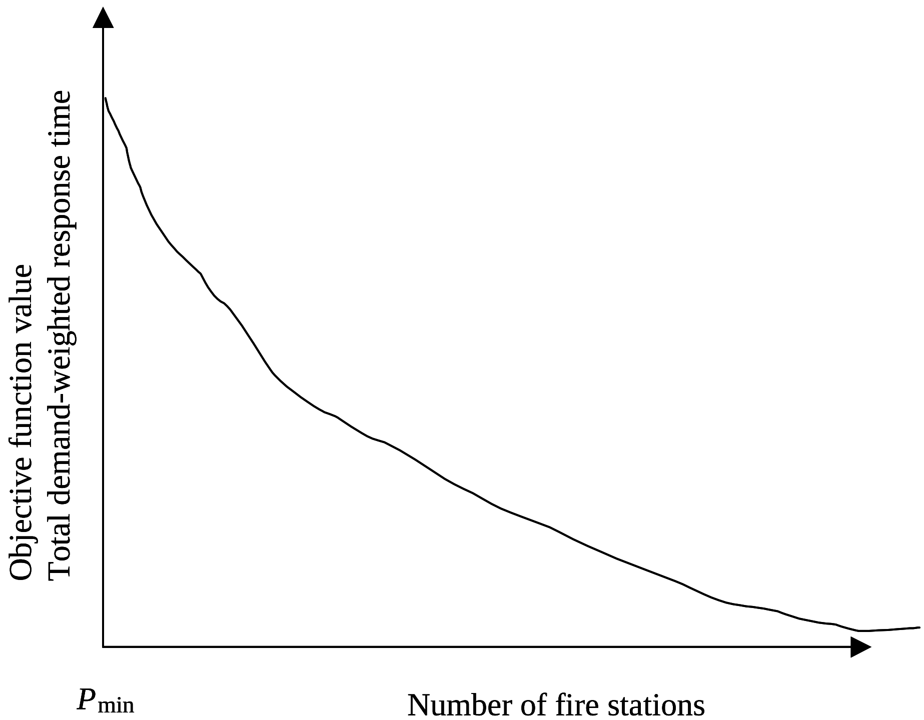
$$X_j \in \{0,1\} \qquad \forall j \in J$$
$$Y_{ij} \in \{0,1\} \qquad \forall i \in I; \forall j \in J$$

g) For a given value of T^c, if the number of fire stations to be located is too small, the problem is not feasible. For example, consider the following two-node problem in which the demand and candidate sites are the two nodes and the value of $T^c = 5$ and $P=1$. How would you go about finding the minimum number of fire stations that need to be located to ensure that the problem is feasible? **Briefly** discuss.



You could solve a set covering problem with a coverage time of T^c to obtain the minimum number of fire stations needed.

h) Let $P_{\min}$ be the minimum number of fire stations that are needed to ensure feasibility. Sketch the objective function value as the number of facilities increases from this value.



Problem 3:

You are concerned with designing a bus route from O'Hare International Airport to downtown Chicago. Passengers arrive at the bus terminal at O'Hare at a constant (non-random) rate of two every minute over the course of the entire 12 hour period each day during which the bus service is in operation. The value of passenger waiting time is $10 per hour per person. Dispatching a bus costs $54 when you account for the cost of the bus, the driver, insurance, fuel, etc.

Define the following general notation:

Inputs

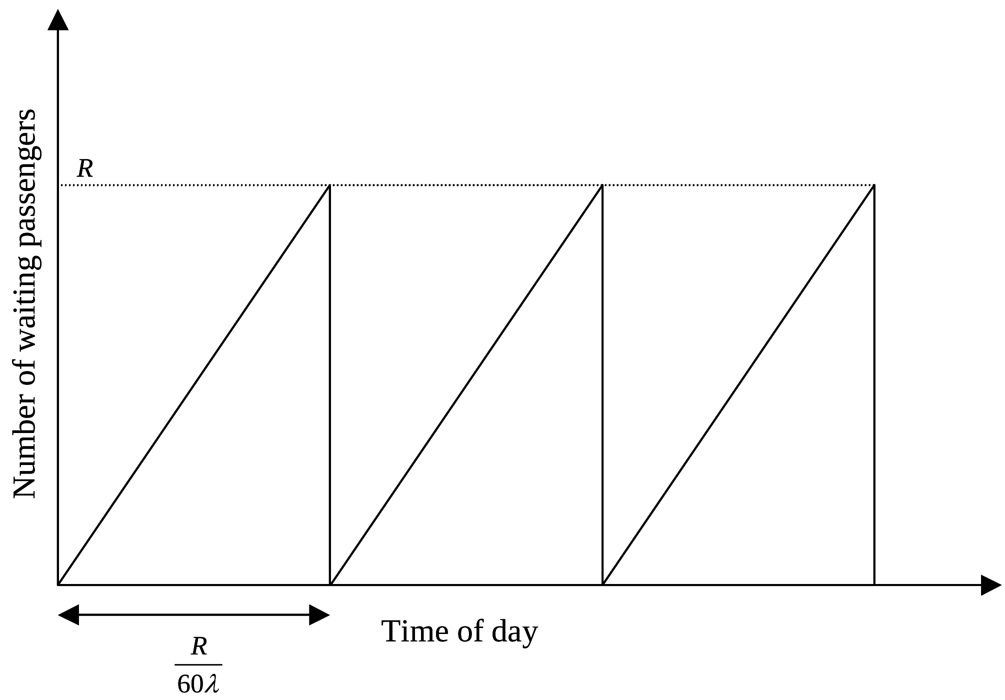
λ	Arrival rate of passengers per minute **(non-random!!)** (this is 2 in our case)
h	number of hours per day that the service operates (this is 12 in our case)
c_1	cost per hour per person of passenger waiting time (this is 10 in our case)
c_2	cost per dispatch (this is 54 in our case)

Decision Variable

R number of passengers on the bus when it departs from O'Hare

Hint: This is not a queuing question.

a) On the diagram below, draw the number of passengers waiting for a bus as a function of the time of day. *Assume that you begin the day with no one waiting.*



b) Write an expression in terms of $\lambda, h,$ and R for the number of bus dispatches per day.

$$\frac{60\lambda h}{R}$$

c) If the bus departs when there are R passengers on the bus (or equivalently waiting for the bus), write an expression for the average passenger waiting time in terms of R and λ.

$$\frac{R}{120\lambda}$$

d) Using your results for the two parts above, write an expression for the total cost (passenger waiting cost and bus dispatch cost) in terms of $c_1, c_2, \lambda, h,$ and R.
$c_1, c_2, \lambda, h,$ and R. *Remember that the answer in (c) is per passenger and you need to account for the total number of passengers served each day.*

$$TC = c_2 \frac{60\lambda h}{R} + c_1 \frac{R}{120\lambda} 60\lambda h = c_2 \frac{60\lambda h}{R} + c_1 \frac{Rh}{2}$$

e) For the values of $c_1, c_2, \lambda, h,$ and R given above, find the optimal number of passengers on board each bus. The optimal number is the number that minimizes the total cost (passenger waiting cost and bus dispatch cost).

$$\frac{dTC}{dR} = -c_2 \frac{60\lambda h}{R^2} + c_1 \frac{h}{2} = 0$$
$$R^2 = \frac{120\lambda c_2}{c_1}$$
$$R = \sqrt{\frac{120\lambda c_2}{c_1}}$$
$$R = \sqrt{\frac{120 \bullet 2 \bullet 54}{10}}$$
$$R = 36$$

f) Find the total daily cost (passenger waiting time and bus dispatch cost) if you operate at the optimal dispatch frequency (with the optimal number of passengers per bus).

$$TC = c_2 \frac{60\lambda h}{R} + c_1 \frac{R}{120\lambda} 60\lambda h = c_2 \frac{60\lambda h}{R} + c_1 \frac{Rh}{2}$$
$$= 54 \frac{60 \bullet 2 \bullet 12}{36} + 10 \frac{36 \bullet 12}{2}$$
$$= 4320$$

g) Management wants to operate one bus every 20 minutes. What is the percentage increase in cost if busses are dispatched with this frequency (i.e., when there are 40 passengers on the bus instead of the optimal number that you found in part (e))?

This means that we will dispatch a bus when there are 40 passengers instead of 36 on the bus. Setting *R=40* in the total cost equation, we get

$$TC = c_2 \frac{60\lambda h}{R} + c_1 \frac{R}{120\lambda} 60\lambda h = c_2 \frac{60\lambda h}{R} + c_1 \frac{Rh}{2}$$
$$= 54 \frac{60 \bullet 2 \bullet 12}{40} + 10 \frac{40 \bullet 12}{2}$$
$$= 4344$$
$$\%\ \text{increase} = \frac{24}{4320} \bullet 100 = 0.5\bar{5}\%$$

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Problem 4 **Hint: This is a queuing question!**

Dr. HoldPlease is concerned because many of her patients have complained of not being able to get through to her office when they try calling and that they are put on hold for a long time when they finally do get through. After collecting data, she finds that patients call her office at a rate of 10 per hour. Furthermore, being a former IE major, she is able to verify that calls arrive according to a Poisson process with this arrival rate. She also finds that the receptionist spends an average of 5 minutes on the phone per patient and that this time is exponentially distributed. She has 3 phone lines that the receptionist manages. In other words, a maximum of 3 patients can be in the phone system at any time (presumably with 1 talking to the receptionist and 2 on hold).

Being a former IE major, she realizes that this is an M/M/1 queue with a finite number allowed in the system. She knows that the equations governing such a queue with S servers (receptionists) and N phone lines are:

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} P_0 & n = 0,1,\ldots,S-1 \\ \left(\frac{\lambda}{\mu}\right)^n \frac{1}{S!}\left(\frac{1}{S}\right)^{n-S} P_0 & n = S, S+1,\ldots,N \end{cases}$$

$$P_0 = \left\{ \sum_{n=0}^{S-1}\left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \sum_{n=S}^{N}\left(\frac{\lambda}{\mu}\right)^n \frac{1}{S!}\left(\frac{1}{S}\right)^{n-S} \right\}^{-1}$$

$$= \begin{cases} \left[\sum_{n=0}^{S-1}\left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \left(\frac{\lambda}{\mu}\right)^S \frac{1}{S!}\left(\frac{1-\rho^{N-S+1}}{1-\rho}\right)\right]^{-1} & \rho \neq 1 \\ \left[\sum_{n=0}^{S-1}\left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \left(\frac{\lambda}{\mu}\right)^S \frac{1}{S!}(N-S+1)\right]^{-1} & \rho = 1 \end{cases}$$

$$\rho = \frac{\lambda}{S\mu}$$

$$L_q = \begin{cases} P_0\left(\frac{\lambda}{\mu}\right)^S \frac{1}{S!}\left[\frac{\rho\left(1-\rho^{N-S}\right)}{(1-\rho)^2} - \frac{(N-S)\rho^{N-S+1}}{1-\rho}\right] & \rho \neq 1 \\ P_0\left(\frac{\lambda}{\mu}\right)^S \frac{1}{S!}\left[\frac{(N-S)(N-S+1)}{2}\right] & \rho = 1 \end{cases}$$

a) Using the values of $\lambda, \mu, S,$ and N given above ($\lambda = 10, \mu = 12, S = 1,$ and $N = 3$) find

1) The probability that the system is empty (P_0)

$$P_0 = \frac{1}{\left[\sum_{n=0}^{S-1}\left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} + \left(\frac{\lambda}{\mu}\right)^S \frac{1}{S!}\left(\frac{1-\rho^{N-S+1}}{1-\rho}\right)\right]}$$
$$= \frac{1}{\left(\frac{5}{6}\right)^0 \frac{1}{0!} + \left(\frac{5}{6}\right)^1 \frac{1}{1!}\left(\frac{1-(5/6)^{3-1+1}}{1-(5/6)}\right)}$$
$$= \frac{1}{1+2.10648}$$
$$= \frac{1}{3.10648}$$
$$= 0.32191$$

2) The probability of being in each other state (P_1, P_2, P_3). Be sure that the sum of the state probabilities equals 1!!

$$P_0 = 0.32191$$
$$P_1 = 0.32191\left(\frac{5}{6}\right) = 0.26826$$
$$P_2 = 0.26826\left(\frac{5}{6}\right) = 0.22355$$
$$P_3 = 0.22355\left(\frac{5}{6}\right) = 0.18629$$

3) The average number of people waiting on hold (L_q). Note that you can verify your work by (a) using the formula above and (b) using $L_q = 2P_3 + P_2$.

$$L_q = P_0\left(\frac{\lambda}{\mu}\right)^S \frac{1}{S!}\left[\frac{\rho\left(1-\rho^{N-S}\right)}{(1-\rho)^2} - \frac{(N-S)\rho^{N-S+1}}{1-\rho}\right]$$
$$= 0.32191\left(\frac{5}{6}\right)^1 \frac{1}{1!}\left[\frac{(5/6)\left(1-(5/6)^2\right)}{(1-(5/6))^2} - \frac{(2)(5/6)^3}{1-(5/6)}\right]$$
$$= 0.59613$$
$$= 2(0.18629) + 0.22355 = 0.59613$$

4) The effective arrival rate ($\lambda_{eff} = \lambda(1 - P_N)$).

$$\begin{aligned}\lambda_{eff} &= \lambda(1 - P_N) \\ &= 10(1 - 0.18629) \\ &= 8.13711\end{aligned}$$

5) The average time spent waiting on hold in minutes (W_q)

$$W_q = \frac{L_q}{\lambda_{eff}} = \frac{0.59613}{8.13711} = 0.07326 \text{ hours} = 4.3956 \text{ or } 4.4 \text{ min}$$

6) The average number of callers per hour who get a busy signal (λP_N)

$$\lambda P_N = 10(0.18629) = 1.8629$$