Notes on Formulation of Optimization Problems

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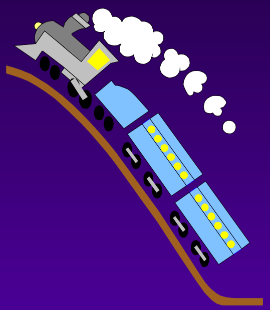
Outline

- Conventions
- Formulation Rules
- Typical Constraint Forms
- Core Models

Key points

- Formulation is an **art** and a **skill** that takes practice – a lot like learning a new language
- Often multiple formulations are valid, though some are better than others



Key points

- **What do you know?**
  - Inputs
- **What do you need to decide?**
  - Decision variables
  - Auxiliary variables (also decision variables)
- **What is/are the goal(s)?**
  - Objective function(s)

Conventions I (generally) Use

- **INDICES**
  - letters like i, j, k, (middle of alphabet)
  - used as subscripts
  - index items in sets with corresponding upper case letter (e.g, I, J, K)
- **INPUTS or DATA**
  - lower case letters near beginning of alphabet
- **PARAMETERS**
  - Like inputs but you may vary these from run to run
- **DECISION VARIABLES**
  - upper case letters near end of alphabet

Indices, Inputs and Dec. Var

- **INDICES**
  - used for enumerating items (e.g., demand nodes, candidate sites, scenarios, time periods, flight legs, production plants, …)
- **INPUTS**
  - you **know** these before you start the problem or can readily compute them from other inputs
  - demand values, distances, costs, plant capacities, number of people required on duty, …

Indices, Inputs, and Dec. Var.

- **DECISION VARIABLES**
  - these are what you want to know or what you must determine **within** the model along the way to determining what you <u>really</u> want to know
  - Production quantities, inventory carryover, shipments between plants and distribution centers, quantities of raw materials to purchase, number of employees to start each period, assignments of students to seminars, …

Objective function

- **OBJECTIVE FUNCTION**
  - this is what you want to minimize or maximize
  - may be a single decision variable
  - more often will be a function of decision variables (e.g., total shipment cost, total production+inventory cost, total number of regular shift and overtime employees, penalties for not assigning to first choice, …)

Formulation rules

- Daskin's 10 (or 11) rules of formulation

Rule 1

- Clearly define all subscripts (at least in your own mind) and sets. For example:
  - I: set of production plants indexed by *i*
  - J: set of demand nodes indexed by *j*
  - T set of time periods indexed by *t*

Rule 2

- Clearly separate the definitions of
  - indices and sets
  - inputs (or parameters)
  - decision variables

Rule 3

- In defining inputs or decision variables in words, if an index appears in the input or decision variable it should appear in the verbal definition as well

d_{ij} = distance between production plant i and demand node j

This one (above) is fine

d_{ij} = distance

This one (above) is BAD

Rule 4

- Do not leave dangling subscripts in the objective function

minimize $$\sum_{i \in I} \sum_{j \in J} c_{ij} X_{ij}$$ Fine

minimize $$\sum_{i \in I} c_{ij} X_{ij}$$ **BAD;**
j index is dangling

Rule 5

- At least some decision variable must appear in the objective function and in each constraint

minimize $\sum_{i \in I} \sum_{j \in J} c_{ij} X_{ij}$ Fine

subject to $d_{ij} \geq 0$ **BAD if d_{ij} is an input distance. No decision variable here**

Rule 6

- Be sure all variables are linked in some way to each other (otherwise the problem is separable and you probably have an error)

Rule 6 example

maximize $\sum_{i \in I} h_i Z_i$

subject to $\sum_{j \in J} X_j = P$

$Z_i \in \{0,1\} \qquad \forall i \in I$

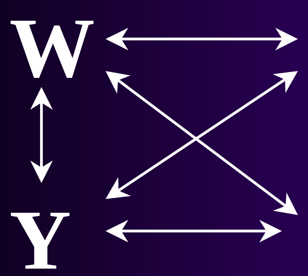
$X_j \in \{0,1\} \qquad \forall j \in J$

X and Z variables are unlinked. You need an additional constraint. e.g.,

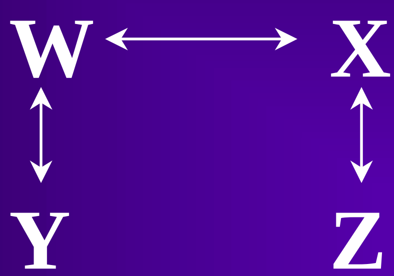
$$Z_i - \sum_j a_{ij} X_j \le 0 \qquad \forall i \in I$$

More on Rule 6

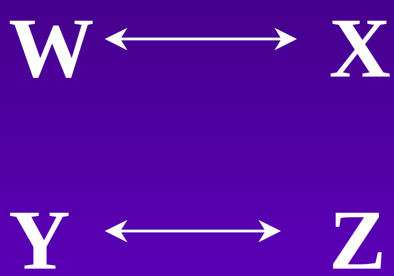
- Each variable does not have to be directly linked to each other variable



ok, but not necessary. may be overconstrained



ok, all variables linked



not ok;
W and X linked;
Y and Z linked;
but (W,X) not linked to (Y,Z)

Rule 7

- If a variable or constant used in a constraint includes some index, then either
  - you should be summing over the index OR
  - you should specify the values of the index to which the constraint applies
  - **DO NOT DO BOTH in the same constraint**

Rule 7 examples

$$\sum_{j \in J} X_{ij} = 1 \qquad \forall i \in I$$

Ok

$$\sum_{j \in J} h_{ik} d_{ijk} Y_{ijk} \le D \qquad \forall i \in I$$

BAD; Need to specify what is going on with index *k*

$$\sum_{j \in J} Y_{ij} = 1 \qquad \forall j \in J$$

BAD; Summing over *j* and specifying constraint applies to all *j*; Also, what is going on with index *i*?

Rule 8 – VERY IMPORTANT

- Try to keep it linear (**IF POSSIBLE**)
  - avoid multiplying decision variables in the objective function or in constraints
  - avoid raising a decision variable to some power
  - avoid logs, trig functions, ...
  - be creative in transformations

Rule 9

- Avoid big M type constraints (**IF POSSIBLE**)
  - constraints with a big value of some constant multiplied by a binary variable
  - often used to turn on or off a constraint depending on the value of the variable
  - may be unavoidable

Rule 10

- Disaggregate constraints when possible

$$Y_{ij} \le X_j \qquad \forall i \in I, \forall j \in J$$

Good, disaggregate constraint

$$\sum_{i \in I} Y_{ij} \le |I| X_j \qquad \forall j \in J$$

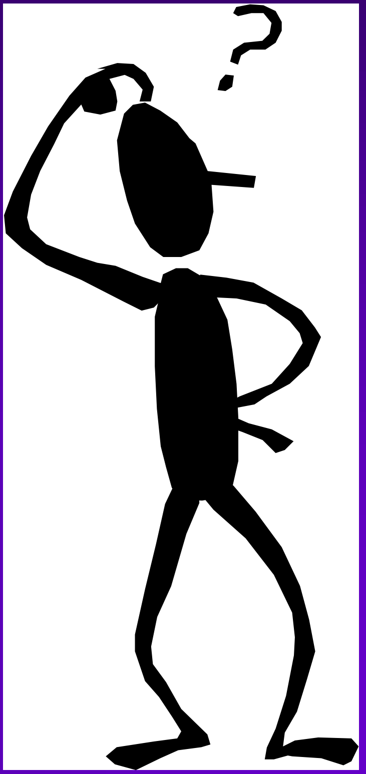
Not so good, aggregate constraint. Will lead to weaker LP relaxations

Rule 11

- Know which of rules 1-10 can be bent and when and how to do so

Multi-Objective and Scenario Planning

1+1=3



Typical Constraint Forms

- TOTAL CONSTRAINT

$$\sum_{j \in J} X_j = p$$

- the total of all the X_j variables must be p
- e.g., Pick p of the X_j variables and set them to 1, set all others to 0 (for X_j a binary variable)

Typical Constraint Forms

- SELECTION or ASSIGNMENT CONSTRAINT

$$\sum_{j \in J} Y_{ij} = 1 \qquad \forall i \in I$$

- For each row *i* (e.g., each demand node), the total of the Y_{ij} variables (for that *i*) must be 1
- each node *i* must be assigned to exactly one facility node

Typical Constraint Forms

- SELECTION or ASSIGNMENT CONSTRAINT

$$\sum_{k \in K_j} X_{jk} \le 1 \quad \forall j \in J$$

- pick at most one capacity for each site j (where K_j is a set of available capacities at candidate site j)
- Note that if the left hand side is 0, it simply means we do not build at candidate site j

Typical Constraint Forms

- SUPPLY Constraints

$$\sum_{j \in J} X_{ij} \leq S_i \quad \forall i \in I$$

where

X_{ij} = flow from i to j

- The total flow out of node *i* must be less than or equal to the supply at node *i* (S_i)
- **Note the definition of X_{ij} and that *i* is being used as a supply node and *j* is being used as a demand node <u>in this case</u>**

Typical Constraint Forms

- DEMAND Constraints

$$\sum_{i \in I} X_{ij} \geq D_j \qquad \forall j \in J$$

where

X_{ij} = flow from i to j

- The total flow into node *j* must be greater than or equal to the demand at node *j* (D_j)
- **Note the definition of X_{ij} and that *i* is being used as a supply node and *j* is being used as a demand node <u>in this case</u>**

Typical Constraint Forms

- DEMAND-LIKE Constraints

$$\sum_{k \in K} q_k Z_k \geq \alpha$$

- Total probability of selected scenarios must be at least α where K is a set of scenarios and q_k is the probability associated with scenario k
- Used in α-reliable minimax regret model in which you have to plan for enough scenarios to get a total planned-for probability of α

Typical Constraint Forms

- LINKAGE or FORCING CONSTRAINTS

$$Y_{ij} \le X_j \qquad \forall i \in I, \forall j \in J$$

- X_j must be at least as large as Y_{ij} OR
- Y_{ij} must be no bigger than X_j for each pair of *i* and *j*
- You cannot assign demands at *i* to a facility at *j* (Y_{ij}=1) unless you locate at *j* (X_j=1)

Typical Constraint Forms

- LINKAGE or FORCING CONSTRAINTS

$$Z_i - \sum_{j \in J} a_{ij} X_j \le 0 \qquad \forall i \in I$$

- Node *i* cannot be counted as being covered (Z_i=1) unless there is at least one facility that is located that is capable of covering node *i* ($\sum_{j \in J} a_{ij} X_j \ge 1$)

Typical Constraint Forms

- LARGEST OF Constraints

$$W \geq \sum_{j \in J} d_{ij} Y_{ij} \qquad \forall i \in I$$

- W must be larger than the largest value of $\sum_{j \in J} d_{ij} Y_{ij}$ for every i
- Typically $\sum_{j \in J} d_{ij} Y_{ij}$ would represent the distance between node i and the facility to which it is assigned
- Used in P-center problems in which we minimize W subject to this and other constraints

Typical Constraint Forms

- CONSTRAINTS THAT SWITCH ON and OFF

$$R \geq V_k - \hat{V}_k - M\left(1 - Z_k\right) \qquad \forall k \in K$$

where

$$M \qquad = \qquad \text{a very large number}$$

so

$$\text{if } Z_k = 1 \text{ then } R \geq V_k - \hat{V}_k$$

$$\text{but if } Z_k = 0 \text{ then } R \geq V_k - \hat{V}_k - M$$

$$\text{and constraint is "inactive"}$$

Typical Constraint Forms

- CONSTRAINTS THAT SWITCH ON and OFF
  - If $Z_k=1$ then constraint is active, otherwise it is "inactive"
  - Used in α-reliable minimax regret model
  - Note that in this case, the remainder of the constraint (without the term in *M*) is a LARGEST OF constraint
  - Try to avoid big-M constraints (see rule 9)

Typical (basic) Models

- Covering based models
  - Set covering
  - Maximal covering
  - P-center
- Average distance based models
  - P-median
  - Fixed charge model

Do not confuse covering and average distance models

Multi-Objective and Scenario Planning

Set Covering Model

minimize $$\sum_{j \in J} X_j$$ Number selected

subject to $$\sum_{j \in J} a_{ij} X_j \geq 1 \quad \forall i \in I$$ Demand-like constraint

$$X_j \in \{0,1\} \qquad \forall j \in J$$ Integrality

Maximal Covering Model

maximize	$\sum_{i \in I} h_i Y_i$		Number Covered
subject to	$\sum_{j \in J} a_{ij} X_j \geq Y_i$	$\forall i \in I$	Coverage Constraint (linkage)
	$\sum_{j \in J} X_j = p$		Number to Locate
	$X_j \in \{0,1\}$	$\forall j \in J$	Integrality
	$Y_i \in \{0,1\}$	$\forall i \in I$	

P-center Model

minimize	W		Maximum Distance
subject to	✓ $\sum_{j \in J} Y_{ij} = 1$	$\forall i \in I$	ASSIGNMENT constraint
	✓ $\sum_{j \in J} X_j = p$		TOTAL constraint
	✓ $Y_{ij} - X_j \le 0$	$\forall i \in I, \forall j \in J$	LINKAGE constraint
	$W \ge \sum_{j \in J} d_{ij} Y_{ij}$	$\forall i \in I$	MAXIMUM constraint
	✓ $X_j \in \{0,1\}$	$\forall j \in J$	INTEGRALITY
	✓ $Y_{ij} \in \{0,1\}$	$\forall i \in I, \forall j \in J$	

✓ → **Same as P-median model (next slide)**

P-median Model

minimize $\sum_{i \in I} \sum_{j \in J} h_i d_{ij} Y_{ij}$ — Demand Wtd Total Dist

subject to

$$\sum_{j \in J} Y_{ij} = 1 \qquad \forall i \in I$$

ASSIGNMENT constraint

$$\sum_{j \in J} X_j = p$$

TOTAL constraint

$$Y_{ij} - X_j \le 0 \qquad \forall i \in I, \forall j \in J$$

LINKAGE constraint

$$X_j \in \{0,1\} \qquad \forall j \in J$$

$$Y_{ij} \in \{0,1\} \qquad \forall i \in I, \forall j \in J$$

INTEGRALITY

Fixed Charge Loc. Model

minimize $\sum_{j \in J} f_j X_j + \beta \sum_{i \in I} \sum_{j \in J} h_i d_{ij} Y_{ij}$ Fixed + Transport Cost

subject to $\sum_{j \in J} Y_{ij} = 1 \quad \forall i \in I$ ASSIGNMENT constraint

$\sum_{j \in J} X_j = p$ TOTAL constraint

$Y_{ij} - X_j \le 0 \quad \forall i \in I, \forall j \in J$ LINKAGE constraint

$X_j \in \{0,1\} \quad \forall j \in J$

INTEGRALITY

$Y_{ij} \in \{0,1\} \quad \forall i \in I, \forall j \in J$