**<u>Question</u>**: How large should *n* (the sample size) be to attain a given power $1-\beta$ and a given value of $\alpha$, the Probability of a type I error if we want to detect a specific difference $\delta$ from $\mu_0$?

Suppose we are considering the following hypothesis testing environment:

$$H_0: \quad \mu = \mu_0$$
$$H_1: \quad \mu \neq \mu_0$$

for some value of $\alpha$.

We then have:

$$\begin{aligned}
Power(\mu_1) &= P(\text{reject } \mathrm{H}_0 | \mu_1) \\
&= P\left( \overline{X} < \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \middle| \mu_1 \right) + P\left( \overline{X} > \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \middle| \mu_1 \right) \\
&= P\left( \frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} < \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} - z_{\alpha/2} \right) + P\left( \frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}} > \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} + z_{\alpha/2} \right) \\
&= P\left( Z < \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} - z_{\alpha/2} \right) + P\left( Z > \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} + z_{\alpha/2} \right)
\end{aligned}$$

The first equality is the definition of the power. The second comes from defining the region in which we would reject the null hypothesis for the two-sided hypothesis test. This is simply when $\overline{X} < \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ or $\overline{X} > \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. The third equality comes from subtracting the mean under the alternate hypothesis ($\mu_1$) and dividing by the standard deviation of the sample mean under the alternate hypothesis ($\sigma/\sqrt{n}$). The fourth equality comes from realizing that, under the alternate hypothesis, the quantity $\frac{\overline{X} - \mu_1}{\sigma/\sqrt{n}}$ is a standard Normal variate.

Now suppose that $\mu_1 = \mu_0 + \delta$ or $\mu_0 - \mu_1 = -\delta$. We then obtain

$$
\begin{aligned}
Power\left(\mu_1\right) &= P\left(Z < \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) + P\left(Z > \frac{\mu_0 - \mu_1}{\sigma/\sqrt{n}} + z_{\alpha/2}\right) \\
&= P\left(Z < \frac{-\delta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) + P\left(Z > \frac{-\delta}{\sigma/\sqrt{n}} + z_{\alpha/2}\right) \\
&= P\left(Z < \frac{-\delta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) + P\left(Z < \frac{\delta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right)
\end{aligned}
$$

The first equality is just a repeat of the final equality above. The second comes from substituting $\mu_0 - \mu_1 = -\delta$ into the probability statements in the first line. The third comes from realizing that for the normal distribution, the probability that a standard Normal random variable exceeds any value $\theta$ is the same as the probability that the standard Normal random variable is less than $-\theta$.

Now we can approximate this by noting that

if $\delta > 0$ then $P\left(Z < \frac{-\delta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) \approx 0$

or if $\delta < 0$ then $P\left(Z < \frac{\delta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) \approx 0$

So for $\delta > 0$, we have

$$
\begin{aligned}
Power\left(\mu_1\right) &= P\left(Z < \frac{-\delta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) + P\left(Z < \frac{\delta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right) \\
&\approx P\left(Z < \frac{\delta}{\sigma/\sqrt{n}} - z_{\alpha/2}\right)
\end{aligned}
$$

But we also know that the $Power = 1 - \beta = \Phi\left(z_\beta\right)$ where $\Phi(x)$ is the cumulative standard Normal distribution evaluated from $-\infty$ up to $x$, or the probability that a standard Normal random variable is less than or equal to $x$. Equating the two terms in the statements of the power we have

$$z_\beta = \frac{\delta}{\sigma/\sqrt{n}} - z_{\alpha/2}$$

$$\frac{\delta}{\sigma/\sqrt{n}} = z_\beta + z_{\alpha/2}$$

$$\frac{\delta\sqrt{n}}{\sigma} = z_\beta + z_{\alpha/2}$$

$$\sqrt{n} = \frac{(z_\beta + z_{\alpha/2})\sigma}{\delta}$$

$$n = \left[\frac{(z_\beta + z_{\alpha/2})\sigma}{\delta}\right]^2$$

Thus, we find that the sample size goes up as

- The variability of the underlying data increases ($\sigma$ goes up)
- The magnitude of the difference that we want to detect goes down ($\delta$ decreases).
- The probability of a type I error (rejecting the null hypothesis when it is in fact true) goes down (as $\alpha$ goes down, $z_{\alpha/2}$ goes up)
- The power (probability of rejecting the null hypothesis when it is wrong) goes up. The power goes up when $\beta$ goes down, which means that $z_\beta$ goes up.

To give you a feel for how fast *n* can increase, we have the following table for $\sigma = 10$ and $\delta = 5$

| | **beta** | | | |
|---|---|---|---|---|
| **alpha** | **0.2** | **0.1** | **0.05** | **0.01** |
| **0.1** | 25 | 35 | 44 | 64 |
| **0.05** | 32 | 43 | 52 | 74 |
| **0.01** | 47 | 60 | 72 | 97 |
| **0.001** | 69 | 84 | 98 | 127 |

For $\sigma = 10$ and $\delta = 2$, we get values that are roughly 6.25 times as big (since $\delta$ decreased by a factor of 2.5).

| | **beta** | | | |
|---|---|---|---|---|
| **alpha** | **0.2** | **0.1** | **0.05** | **0.01** |
| **0.1** | 155 | 215 | 271 | 395 |
| **0.05** | 197 | 263 | 325 | 460 |
| **0.01** | 292 | 372 | 446 | 601 |
| **0.001** | 427 | 523 | 609 | 789 |

Finally, for $\sigma = 10$ and $\delta = 1$, we get

| | beta | | | |
|---|---|---|---|---|
| **alpha** | **0.2** | **0.1** | **0.05** | **0.01** |
| **0.1** | 619 | 857 | 1083 | 1578 |
| **0.05** | 785 | 1051 | 1300 | 1838 |
| **0.01** | 1168 | 1488 | 1782 | 2404 |
| **0.001** | 1708 | 2091 | 2436 | 3155 |

**(Note that what is important is the ratio of** $\frac{\sigma}{\delta}$ **and not their absolute values. Thus, we would get the same results as those shown in the first table if we had used** $\sigma = 2$ and $\delta = 1$**, for example.)**