Outline

1. How to Convexify Nonconvex QP
2. Continuous Case
3. Mixed Binary Case
4. Mixed Integer Case
5. Final Thoughts
How to Convexify Nonconvex QP
\[
\min \quad c^T x \\
\text{s.t.} \quad x \in F
\]

\[
F := \left\{ x \in \mathbb{R}^n : \begin{array}{l} Ax \leq b \\ x_j \in \mathbb{Z} \quad \forall \ j \in J \end{array} \right\}
\]

What is \( \text{conv}(F) \)?
\begin{align*}
\min & \quad x^T Q x + 2 c^T x \\
\text{s.t.} & \quad x \in F
\end{align*}
\downarrow
\begin{align*}
\min & \quad \begin{pmatrix} 0 & c^T \\ c & Q \end{pmatrix} \cdot \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} \\
\text{s.t.} & \quad x \in F
\end{align*}
\[
\min \begin{pmatrix} 0 & c^T \\ c & Q \end{pmatrix} \cdot Y \\
\text{s.t.} \quad Y \in \hat{F}
\]

\[
\hat{F} := \left\{ \begin{pmatrix} 1 & x^T \\ x & xx^T \end{pmatrix} : x \in F \right\}
\]
What is $HG := \text{conv}(\hat{F})$?


- $HG$ is not polyhedral, and taking the closure is necessary
- This approach may not be the most practical approach
  - Certainly characterizing $HG$ is a hard problem
- If your QP doesn't contain all $O(n^2)$ variables, it may make sense to convexify just over your variables
  - In fact, it might be better to lift to an even larger set of variables
- But understanding $HG \subseteq \mathbb{S}^{n+1}$ is our goal today
Observation. Fully characterizing $HG$ is equivalent to identifying all quadratics $x^T Q x + 2 c^T x + \kappa$, which are nonnegative for all $x \in F$

Which is equivalent to identifying all matrices

$$
\begin{pmatrix}
\kappa & c^T \\
c & Q
\end{pmatrix} \in S^{n+1}
$$

whose associated quadratic is nonnegative over $F$. Such matrices are called copositive over $F$

In fact, $HG$ is essentially the dual of "copositive over $F"
Three Types of Valid Inequalities
• Given $F$, we will identify valid inequalities for $HG$

• Each will come from one of three classes...
Type 1: Explicit Quadratics

If

\[ x^T Q x + 2c^T x + \kappa \geq 0 \]

constrains \( F \), then

\[ Q \cdot X + 2c^T x + \kappa \geq 0 \]

is a valid linear inequality for \( HG \)
Explicit quadratic for the complement of an ellipsoid
Type 2: Linear Conjunctions

If

\[ a_1^T x + b_1 \geq 0 \quad \text{and} \quad a_2^T x + b_2 \geq 0 \]

are valid for \( F \), then

\[ \frac{1}{2} (a_1 a_2^T + a_2 a_1^T) \cdot X + (b_2 a_1 + b_1 a_2)^T x + b_1 b_2 \geq 0 \]

is a valid linear inequality for \( HG \)
Linear conjunction
Type 3: Linear Disjunctions

If every $x \in F$ satisfies

$$\text{either } a_1^T x + b_1 \geq 0 \quad \text{or} \quad a_2^T x + b_2 \geq 0$$

then

$$\frac{1}{2}(a_1 a_2^T + a_2 a_1^T) \cdot X + (b_2 a_1 + b_1 a_2)^T x + b_1 b_2 \leq 0$$

is a valid linear inequality for $HG$
Continuous Nonconvex QP
1. Unconstrained
2. Linear equations
3. Nonnegative orthant
4. Nonnegative orthant, linear equations, and complementarities
5. Linear inequalities
6. Half-ellipsoid
7. Swiss cheese
Unconstrained
\[ F = \mathbb{R}^n \]

\[ H_G = \left\{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} : X \succeq xx^T \right\} =: PSD \]
Proposition. $HG$ is generated by all disjunctions of the form

\[
\text{either} \quad a^T x + b \geq 0 \quad \text{or} \quad a^T x + b \leq 0
\]

where $(a, b) \in \mathbb{R}^n \times \mathbb{R}$.
Linear Equations
\[ F = \{ x \in \mathbb{R}^n : Ax = b \} \]

\[ \Downarrow \]

\[ HG = PSD \cap \left\{ \begin{array}{c}
Ax = b \\
\text{diag}(AXA^T) = b \circ b
\end{array} \right\} \]
Theorem (B 2009). $HG$ is generated by

1. all PSD disjunctions

2. conjunctions of the form $(a_i^T x = b_i \text{ and } a_j^T x = b_j)$ for all pairs of constraints in $Ax = b$. 
Nonnegative Orthant
\[ F = \mathbb{R}_+^n \]

\[ \Downarrow \]

\[ HG \subseteq PSD \cap \left \{ \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \geq 0 \right \} =: DNN \]
Theorem (Maxfield-Minc 1962). $HG = DNN$ if and only if $n \leq 3$. 
• In the literature, "copositive over the nonnegative orthant" is typically just called "copositive"

• Regarding $F = \mathbb{R}_+^n$, $HG$ is the dual cone of the copositive matrices, i.e., $HG = COP^*$

• Generating a valid inequality for $HG = COP^*$ is hard, but we can do better with a sums-of-squares approach
Nonnegative Orthant, Linear Equations, and Complementarities
\[ F = \left\{ x \geq 0 : \quad Ax = b \quad \text{and} \quad x_j x_k = 0 \quad \forall (j, k) \in E \right\} \]

\[ \Downarrow \]

\[ HG = COP^* \cap \left\{ \begin{array}{l}
Ax = b \\
\text{diag}(AXA^T) = b \circ b \\
X_{jk} = 0 \quad \forall (j, k) \in E
\end{array} \right\} \]
Linear Inequalities
\[ F = \{ x \in \mathbb{R}^n : A x \leq b \} \]

\[ \Downarrow \]

\[ H G \subseteq PSD \cap \left\{ (b - A) \begin{pmatrix} 1 & x^T \\ x & X \end{pmatrix} \begin{pmatrix} b^T \\ -A^T \end{pmatrix} \geq 0 \right\} \]

(equality when \text{length}(b) \leq 4)
\[ F = [0, 1]^2 \]

\[ \Downarrow \]

\[ HG = \left\{ \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & X_{11} & X_{12} \\ x_2 & X_{12} & X_{22} \end{pmatrix} \succeq 0 : \begin{align*} X_{11} & \leq x_1, \ X_{22} \leq x_2 \\ X_{12} & \leq x_1 \\ X_{12} & \leq x_2 \\ X_{12} & \geq 0 \\ X_{12} & \geq x_1 + x_2 - 1 \end{align*} \right\} \]
The linear constraints

\[
\begin{pmatrix}
  b \\
  x \\
  X
\end{pmatrix}
\begin{pmatrix}
  1 & x^T \\
  1 & x \\
  1 & 1
\end{pmatrix}
\begin{pmatrix}
  b^T \\
  X^T \\
  -A^T
\end{pmatrix} \geq 0
\]

are known as *RLT constraints*
Half-Ellipsoid
\[ F = \{ \| x \| \leq 1 : a^T x \leq b \} \]

\[ \Downarrow \]

\[ HG = PSD \cap \{ \| bx - Xa \| \leq b - a^T x \} \]
The constraint

$$\|bx - Xa\| \leq b - a^T x$$

is known as an \textit{SOCRLT constraint}
Theorem (Yang-Anstreicher-B 2018). Consider the intersection $F$ of

- ("ball") $\|x\| \leq 1$
- ("cuts") $Ax \leq b$
- ("holes") $x^T Q_k x + 2c_k^T x + \kappa_k \geq 0$ for all $k$, where each $Q_k \succ 0$

If none of the cuts and holes touch each other, then

$$HG = PSD \cap RLT \cap SOCRLT \cap \{Q_k \bullet X + 2c_k^T x + \kappa_k \geq 0\}$$
Mixed Binary QP
\[ F = \left\{ x \geq 0 : x_j x_k = 0 \forall (j, k) \in E \right\} \]
\[ x_j \in \{0, 1\} \forall j \in J \]

\[ \downarrow^* \]

\[ HG = COP^* \cap \left\{ \begin{align*}
Ax &= b \\
\text{diag}(AXA^T) &= b \circ b \\
X_{jk} &= 0 \forall (j, k) \in E \\
X_{jj} &= x_j \forall j \in J
\end{align*} \right\} \]
*As long as \( \{ x \geq 0 : Ax = b \} \) ensures

\[
\begin{align*}
&\cdot x_j \leq 1 \text{ for all } j \in J \\
&\cdot x_j, x_k \text{ bounded for all } (j, k) \in E
\end{align*}
\]
$[0, 1]^n$ and $\{0, 1\}^n$
**Proposition.** Since \( \{0, 1\}^n \subset [0, 1]^n \),

\[
HG(\{0, 1\}^n) \subset HG([0, 1]^n)
\]

I.e., any valid inequality for \( HG([0, 1]^n) \) is automatically valid for \( HG(\{0, 1\}^n) \)
A partial converse holds...

**Theorem (B-Letchford 2009).** Adding \( \text{diag}(X) = x \) to \( HG([0, 1]^n) \) captures \( HG(\{0, 1\}^n) \)

**Corollary.** Any valid inequality for \( HG(\{0, 1\}^n) \)—which has no \( X_{jj} \) terms—is automatically valid for \( HG([0, 1]^n) \)
Mixed Integer QP
Integer Lattice
Theorem (B-Letchford 2014). For $F = \mathbb{Z}^n$, \[ HG \subseteq SPLIT \subsetneq PSD \]

The first inclusion holds with equality if $n \leq 2$ but is strict for $n \geq 6$

Note. The cases $n \in \{3, 4, 5\}$ are unresolved
Nonnegative Integer Lattice
Theorem (B-Letchford 2014). For $F = \mathbb{Z}_+^n$, 

$$HG \subseteq \text{SPLIT} \cap \text{SPLIT}_+ \cap \text{RLT}$$

But not much else is known 😞
Remark. Buchheim-Traversi showed how, in practice, to separate:

- $SPLIT$
- $SPLIT_+$ for the ternary case

They also demonstrated the effectiveness of these cuts in terms of closing the gap.
**Crazy Observation.** For both cases $F = \mathbb{Z}^n$ and $F = \mathbb{Z}_+^n$, every extreme point of $HG$ lies on a countably infinite number of facets
Final Thoughts
• For nonconvex QP, there is still lots to do, especially the integer case

• Convexification involves aspects of
  • convex analysis
  • polyhedral theory
  • SDP
  • polynomial optimization

• Hence, a very interesting area to study
Thank You

And good luck with your research!